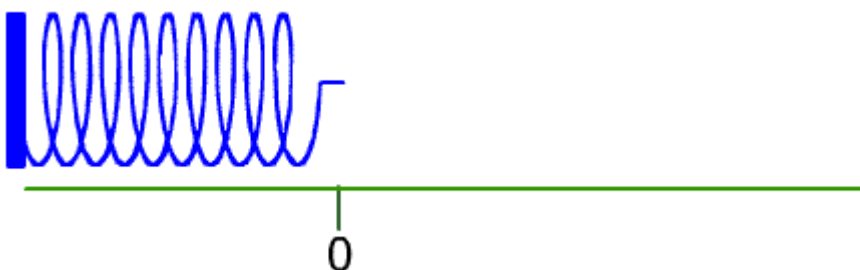


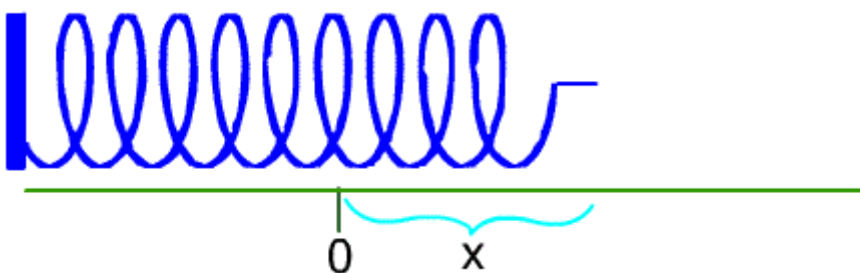
WORK

Definition. Let F be a continuous function representing force on the closed interval $[a, b]$. The work in moving a particle from $x = a$ to $x = b$ along a straight line is the integral

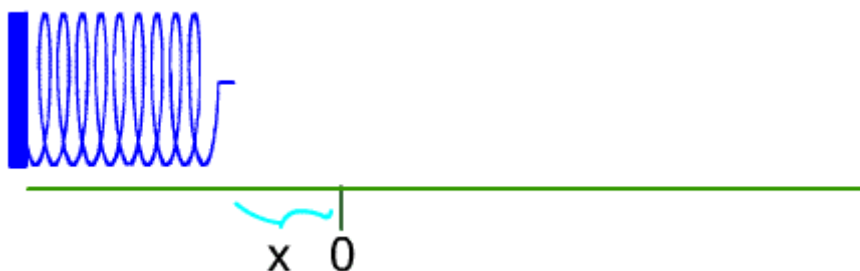
$$\int_a^b F(x) dx.$$



Suppose that we have a spring of length s with a fixed left end. We will set up coordinates so that $x = 0$ occurs at the right end of the spring at rest.



When $x > 0$, then the spring has been stretched x units.
When $x < 0$, then the spring has been compressed x units.



EXAMPLE

Suppose that a spring has a natural length of 2 feet and that a force of 20 pounds is needed to compress the spring to a length of 18 inches.

Find the amount of work that is necessary to stretch the spring from a length of 2.5 feet to a length of 3 feet.

By Hooke's Law, the force needed to stretch a spring a distance of x units is $F(x) = kx$.

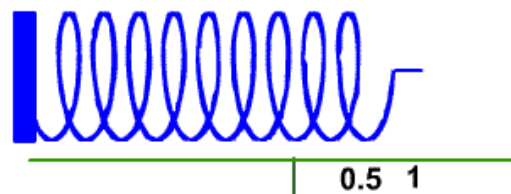
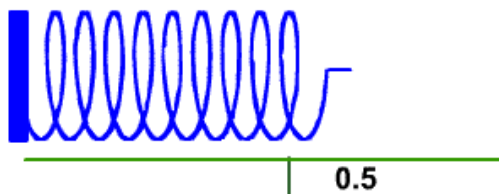
We need to find k . The constant k will vary from spring to spring. For this spring, we know that a compression force of 20 lb will move the spring 6 inches, or $\frac{1}{2}$ ft, to the left. [Remember that at rest $x = 0$.]

We substitute in $F = -20$ lb (negative for compression) and $x = -\frac{1}{2}$ ft, and solve for k :

$$\begin{aligned} -20 &= k(-\frac{1}{2}) \\ 40 &= k \end{aligned}$$

Our force formula for this spring is $F(x) = 40x$ lbs.

Now we want to find the work that is necessary to stretch the spring from a length of 2.5 ft to 3 ft. Since its length at rest is 2 ft, this means that we want $x = .5$ ft to $x = 1$ ft.



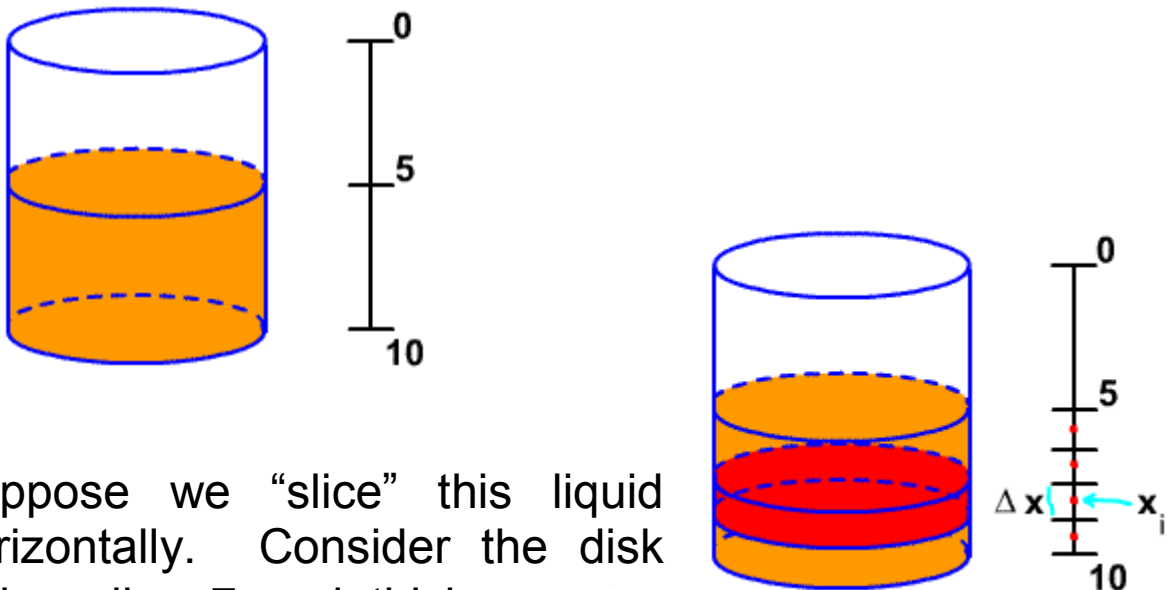
By definition of work, the work is the integral

$$\int_{0.5}^1 40x \, dx = 20x^2 \Big|_{0.5}^1 = 20(1^2 - (0.5)^2) = 15 \text{ ft-lbs.}$$

Example

Suppose that a cylindrical tank has height 10, the radius of the base is 7, and it is half filled with water. Find the amount of work necessary to move all of the water out of the top of the tank.

We set up coordinates as shown.



Suppose we “slice” this liquid horizontally. Consider the disk with radius 7 and thickness Δx .

The volume of this disk is its cross-sectional area (a circle) times its thickness: $\pi r^2 dx = 49\pi dx$. Since water weighs 62.5 pounds per cubic foot, the weight of the water in this disk is $62.5(49)\pi dx$ pounds.

We must move this slice of water up and out of the tank. x_i is the distance the water must be moved. Since work is distance times weight, the work that must be done to move this slice of water is $62.5(49)\pi x_i dx$ foot-pounds.