Vector Methods in Three-Space

This worksheet contains examples of the following vector methods in space:

Finding the distance between two points.

Finding the point that divides a straight line between two points into a given ratio.

Finding the distance from a point to a straight line.

Finding the distance from a point to a plane.

Finding the distance between two parallel planes.

Finding two parallel planes that contain two skew lines.

Finding the distance between two skew lines.

Finding the equation of the line common to two intersecting planes.

Finding the equation of a line through a point and parallel to a vector.

Finding the equation of a plane through a point and perpendicular to a vector.

Finding the equation of a plane through a point and parallel to a vector.

Finding the equation of a plane containing three points.

We generally use three important properties of vectors in three-space to solve many of the problems which arise with points, lines, angles, and planes. These are:

a. If two vectors a, b are parallel, then one is a multiple of the other, i.e. $\langle a \rangle = k \langle b \rangle$ and their cross product is equal to zero, i.e. $\langle a \rangle \times \langle b \rangle = 0$.

b. If two vectors a, b are orthogonal (perpendicular to each other), then their dot product is equal to zero, i.e. $\langle a\rangle \cdot \langle b\rangle = 0$

c. A vector c which is equal to the cross product of two vectors a and b is perpendicular to both a and b, i.e., if $\langle c\rangle = \langle a\rangle \times \langle b\rangle$ then c is not only perpendicular to a, but also perpendicular to b.

Find the distance between two points.

Let the two points be $P = (x_2, y_2, z_2)$ *and* $Q = (x_1, y_1, z_1)$

Then $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

or
$$d = \langle P - Q\rangle$$
$$d^2 = d \cdot d = \langle P - Q\rangle \cdot \langle P - Q\rangle$$
$$d = \sqrt{\langle P - Q\rangle \cdot \langle P - Q\rangle}$$

Example: P=(3,-1,4), Q=(5,4,1)

$$d = \sqrt{(5-3)^2 + (4+1)^2 + (1-4)^2} = \sqrt{4 + 25 + 9} = \sqrt{38}$$

or $\langle P - Q\rangle = \langle 2,5,-3\rangle$

$$d = \sqrt{\langle P - Q\rangle \cdot \langle P - Q\rangle} = \sqrt{\langle 2,5,-3\rangle \cdot \langle 2,5,-3\rangle} = \sqrt{4 + 25 + 9} = \sqrt{38}$$

Find the point that divides a straight line between two points into a given ratio.

Let the position vectors for the two given points on the line be a, b. Let c be the position vector for the point that divides the line into a ratio of m, n.

Then c = (an + bm)/(m + n)

Example: Find the point on the line between (2,-3,7) and (6,4,2) that divides the line into a ratio of 2 to 3.

a = (2,-3,7) b = (6,4,2) m = 2, n = 3

c = [3(2,-3,7) + 2(6,4,2)]/(2 + 3)= (18,1,25)/5

c = (18/5, 1/5, 5)

Find the distance from a point to a straight line.

Let the equation of the straight line be ax + by + c + 0 and let the point be (x_1, y_1),

then the distance is $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ or $d = \frac{\|PQ \times u\|}{\|u\|}$

Where P is the point, Q is any point on the line, and u is the position vector for the line.

Example: What is the distance from the point (3,1) to the line 2x – 4y + 5 = 0? What is the distance from the origin to the line?

$$d = \frac{|2(3) - 4(1) + 5|}{\sqrt{2^2 + 4^2}} = \frac{|6 - 4 + 5|}{\sqrt{20}} = \frac{7}{2\sqrt{5}} = \frac{7\sqrt{5}}{10}$$

or $\mathrm{P} = (3,1),\ \mathrm{Q} = \left(0, \frac{5}{4}\right)\ \mathrm{PQ} = \left\langle 3, -\frac{1}{4}\right\rangle\ \mathrm{u} = \left\langle \frac{1}{2}, \frac{1}{4}\right\rangle$

$$\mathrm{d} = \frac{\|\langle \mathrm{PQ}\rangle \times \mathrm{u}\|}{\|\mathrm{u}\|} = \frac{\left\| \left\langle 3, -\frac{1}{4}\right\rangle \times \left\langle \frac{1}{2}, \frac{1}{4}\right\rangle \right\|}{\frac{\sqrt{5}}{4}} = \frac{7\sqrt{5}}{10}$$

The distance from the origin is $\frac{5}{2\sqrt{5}} = \frac{5\sqrt{5}}{10}$

Find the distance from a point P to a plane.

Let the equation of the plane be ax + by + cz + d = 0 and let the point be

$P = (x_1, y_1, z_1)$. Then the distance is

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad \text{or} \quad d = \frac{\|\langle PQ \rangle \cdot \langle n \rangle\|}{\|n\|}$$

Where Q is a point on the plane, P is not on the plane, PQ is (P – Q), and n is a normal to the plane.

<u>Example</u>: Find the distance from the point (3,1,5) to the plane x – 3y + 2z – 5 = 0

$$d = \frac{|1(3) - 3(1) + 2(5) - 5|}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{5}{\sqrt{14}} = \frac{5\sqrt{14}}{14}$$

Find the distance between two parallel planes. All we need to do is to pick a point in one of the planes; then use the technique in the previous example.

<u>Example</u>: Find the distance between the parallel planes 10x – 12y + 5z – 8 = 0 and 10x – 12y + 5z + 4 = 0. (1,1,2) is a point in the first plane, so

$$d = \frac{|10(1) - 12(1) + 5(2) + 4|}{\sqrt{10^2 + 12^2 + 5^2}} = \frac{12}{\sqrt{269}}$$

Find two parallel planes that contain two skew lines:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}; \frac{x - x_2}{d} = \frac{y - y_2}{e} = \frac{z - z_2}{f}$$

The vectors $u = \langle a, b, c \rangle$ *and* $v = \langle d, e, f \rangle$ are the position vectors for the lines.

The cross product $u \times v$ is a vector perpendicular to both u and v, that is, perpendicular to both lines. Since the two planes will be parallel, both will have the same normal vector. Moreover, since the two lines will be in the planes, the normal vector will also be perpendicular to the lines. We can now write both equations of the planes.

<u>Example</u>: Find the two parallel planes that contain the following two skew lines:

$$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-6}{-2}; \quad \frac{x+1}{4}=\frac{y+1}{5}=\frac{z-1}{3}$$

$$u=\langle 2,1,-2\rangle, \quad v=\langle 4,5,3\rangle$$

$$u \times v=\langle 13,14,6\rangle \text{ (This is the normal vector)}$$

The two planes are:

$$13(x-1)-14(y+3)+6(z-6)=0$$

$$13(x+1)-14(y+1)+6(z-1)=0$$

or

$$13x-14y+6z-91=0$$

$$13x-14y+6z-7=0$$

Find the distance between two skew lines. First, find the two parallel planes that contain the skew lines. Secondly, find the distance between these two planes which will equal the distance between the two skew lines.

<u>Example</u>: For the two skew lines given above, (0,0,7/6) is a point on the second plane. Then we have

$$d=\frac{\left|13(0)+14(0)+6\left(\frac{7}{6}\right)-91\right|}{\sqrt{13^2+14^2+6^2}}=\frac{84}{\sqrt{401}}$$

Find the equation of the line of intersection of two planes. We have two means for solving this type of problem, algebraic and vector methods. Both are illustrated in the example below.

Example: Find the line common to both planes x + y – 3z = 2 and 2x – y + z = 9

Algebraic: Solve for x and y in terms of z.

$$3x = 2z + 11 \qquad x = 11/3 + 2z/3$$

$$3y = 7z - 5 \qquad y = -5/3 + 7z/3$$

Now let z = t, then x = 11/3 + 2t/3 and y = -5/3 + 7t/3, are the parametric equations of the line.

Vector Method: The vectors $u = \langle 1,1,-3 \rangle$ *and* $v = \langle 2,-1,1 \rangle$ are the normal vectors to the planes and so are orthogonal to the common line. Therefore $u \times v = \langle -2,-7,-3 \rangle$ is a direction vector for the line. If z = 0, x = 11/3 , and y = -5/3, then the equation of the line in parametric form is

$$x = 11/3 - 2t$$

$$y = -5/3 - 7t$$

$$z = -3t$$

Find the equation of the line through a point and parallel to a vector.

Let the point be (x_1, y_1, z_1) and let the vector be $\langle a,b,c \rangle$. Then the equation of the line in parametric form is $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$, or in symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

<u>Example</u>: Find the equation of the line through the point (1,5,2) and parallel to the vector $\langle 2,-2,7 \rangle$.

$$x = 1 + 2t,\ \ y = 5 - 2t,\ \ z = 2 + 7t \quad \text{or}$$

$$\frac{x-1}{2} = \frac{y-5}{-2} = \frac{z-2}{7}$$

Find the equation of a plane through a point and perpendicular to a vector.

Let the point be (x_1, y_1, z_1) and the vector be $\langle a,b,c \rangle$. Since the point will be in the plane, selecting another point (x,y,z) in the plane will give us a vector $\langle x - x_1, y - y_1, z - z_1 \rangle$ which is also in the plane. Since this vector will be perpendicular to the given vector, their dot product will be zero. That is,

$$\langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle a,b,c \rangle = 0$$

This is the equation of the plane.

<u>Example</u>: Find the equation of the plane through the point (1,-2,4) and perpendicular to the vector $\langle 2,4,-4 \rangle$.

$$\langle x - 1, y + 2, z - 4 \rangle \cdot \langle 2,4,-4 \rangle = 0$$

$$2(x - 1) + 4(y + 2) - 4(z - 4) = 0$$

$$x + 2y - 2z + 11 = 0$$

Note that $\langle 1,2,-2\rangle \times \langle 2,4,-4\rangle = 0$ shows that the normal to the plane is parallel to the given vector, which means the plane is perpendicular to the given vector.

Find the equation of a plane through a point and parallel to a vector.

Let the point be (x_1, y_1, z_1) and let the vector be $\langle a,b,c\rangle$. Since the point will be in the plane, selecting another point (x,y,z) in the plane will give us a vector $\langle x - x_1, y - y_1, z - z_1\rangle$ which is also in the plane. Since this vector will be parallel to the given vector, their cross product will be zero. That is,

$$\langle x - x_1, y - y_1, z - z_1\rangle \times \langle a,b,c\rangle = 0$$

This is the equation of the plane.

Example: Find the equation of the plane through the point (1,-2,4) and parallel to the vector $\langle 2,4,-4\rangle$.

$$\langle x - 1, y + 2, z - 4\rangle \times \langle 2,4,-4\rangle = 0$$

$$8(x - 1) - 6(y + 2) - 2(z - 4) = 0$$

$$4x - 3y - z = 12$$

Note that $\langle 4,-3,-1\rangle \cdot \langle 2,4,-4\rangle = 8 - 12 + 4 = 0$ shows that the normal to the plane is perpendicular to the given vector, which means the plane is parallel to the given vector.

Find the equation of a plane containing three points not lying in a straight line.

Let the three points be P,Q, and R. Then $\langle P-Q\rangle\, and\, \langle P-R\rangle$ are two vectors in the plane. The vector $v=\langle P-Q\rangle\times\langle P-R\rangle$ is perpendicular to both vectors and is therefore the normal to the plane. Using v and the point P we can write the equation of the plane.

Example: Find the equation of the plane containing the three points (1,2,3), (2,0,4), (3,3,1). Let P = (1,2,3), Q = (2,0,4), R = (3,3,1).

Then: $\mathrm{P}-\mathrm{Q}=\langle -1,2,-1\rangle$, $\mathrm{P}-\mathrm{R}=\langle -2,1,2\rangle$ and $\mathrm{v}=\langle -1,2,-1\rangle\times\langle -2,1,2\rangle$

$\mathrm{v}=\langle 5,4,3\rangle$

Using P and v, $5(x-1)+4(y-2)+3(z-3)=0$ or $5x+4y+3z=22$