THE CHAIN RULE AND DIFFERENTIABILITY

When we were dealing with a function of one variable we were able to say that the function was differentable as long as it had a first derivative. With a function of several variables we need more information to determine its differentiability than the function's partial derivatives. In fact, it is possible for f(x,y) to have partial derivatives at a point (x_0, y_0), and yet not be differentiable at (x_0, y_0). The conditions the function must meet to be differentiable at (x_0, y_0) are as follows:

a. $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, and

b. there are functions g_1 and g_2 so that as $(\Delta x, \Delta y) \to (0,0)$ we also have $(g_1, g_2) \to (0,0)$ and we can write $\Delta f(x_0, y_0)$ as

c. $\Delta f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + g_1 \Delta x + g_2 \Delta y$

<u>Example</u>: Is the function $f(x,y) = 5x + 3y^2$ differentiable at (x,y)?

First find the increment of f(x,y),

$$\Delta f(x,y) = 5(x + \Delta x) + 3(y + \Delta y)^2 - (5x + 3y^2)$$
$$\Delta f(x,y) = 5\Delta x + 6\Delta y + \Delta y^2$$

Next, identify $f_x(x,y)$ and $f_y(x,y)$, then

$$\Delta f(x,y) = f_x(x,y)\Delta x + f_y(x,y)\Delta y + \Delta y \Delta y$$

Now let $g_1 = 0$ and $g_2 = \Delta y$. Then we have as $(\Delta x, \Delta y) \to (0,0)$, $(g_1, g_2) \to (0,0)$ and f(x,y) is differentiable at every (x,y)

THE CHAIN RULE:

If f(u,v) is a function of the two variables u, v and u and v are functions of the variables x,y , then the partial derivatives of f with respect to x and y are:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x}$$
$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u}\frac{\delta y}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y}$$

<u>Example</u>: $f(u,v) = u + 3v + 5;\ u = x^2 + y^2;\ v = x^2 - xy - y^2$

$$\frac{\delta f}{\delta u} = 1;\ \frac{\delta f}{\delta v} = 3;\ \frac{\delta u}{\delta x} = 2x;\ \frac{\delta u}{\delta y} = 2y;\ \frac{\delta v}{\delta x} = 2x - y;\ \frac{\delta v}{\delta y} = -x - 2y$$

Then
$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x}$$
$$= (1)(2y) + (3)(2x - y) = 8x - 3y$$
$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y}$$
$$= (1)(2y) + (3)(-x - 2y) = -4y - 3x$$

If f(u,v,w) is a function of three variables each of which is a function of x and y, then the chain rule can be extended as follows:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta x}$$
$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta y}$$

Example:

$$f(u,v,w) = uv - w^2 \ , u = x^2 - y^2 \ , v = 2xy \ , w = x^2 - xy + y^2$$

$$\frac{\delta f}{\delta u} = v \ \frac{\delta f}{\delta v} = u \ \frac{\delta f}{\delta w} = -2w \ \frac{\delta u}{\delta x} = 2x \ \frac{\delta u}{\delta y} = -2y \ \frac{\delta v}{\delta x} = 2y \ \frac{\delta v}{\delta y} = 2x \ \frac{\delta w}{\delta x} = 2x - y \ \frac{\delta w}{\delta y} = -x + 2y$$

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta x}$$

$$= (v)(2x) + (u)(2y) + (-2w)(2x - y)$$

$$= y^3 - 6xy^2 + 2x^2y + 2x^2 - 4x^3$$

Example:

$$f(x,y) = x^2 + 2xy \qquad x = rCos\theta \quad y = rSin\theta$$

$$\frac{\delta f}{\delta x} = 2x + 2y \ , \frac{\delta f}{\delta y} = 2x \ , \frac{\delta x}{\delta r} = Cos\theta \ , \frac{\delta x}{\delta \theta} = -rSin\theta \ , \frac{\delta y}{\delta r} = Sin\theta \ , \frac{\delta y}{\delta \theta} = rCos\theta$$

$$\frac{\delta f}{\delta r} = (2x + 2y)Cos\theta + (2x)Sin\theta$$

$$\frac{\delta f}{\delta \theta} = (2x + 2y)(-rSin\theta) + (2x)(rCos\theta)$$

Implicit Differentiation:

Whenever a function f(x,y) = 0 , the chain rule gives us another way of doing implicit differentiation. Let z = f(x,y) = 0, then

$$\frac{dz}{dx} = f_x(x,y)\frac{dx}{dx} + f_y(x,y)\frac{dy}{dx} = 0 \quad \text{from which}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{f_x(x,y)}{f_y(x,y)} \ \text{ provided that } f_y(x,y) \neq 0$$

Example: Find $\frac{\text{dy}}{\text{dx}}$ if $x^3 - 3xy^3 + x^2y^2 + 7 = 0$

$f_x = 3x^2 - 3y^3 + 2xy^2$

$f_y = -9xy^2 + 2x^2y$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{3x^2 - 3y^3 + 2xy^2}{9xy^2 - 2x^2y}$$

Suppose that F(x,y,z) = 0, then

$$\frac{\delta z}{\delta x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \qquad \frac{\delta z}{\delta y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} \quad \text{provided that } F_z(x,y,z) \neq 0$$

<u>Example</u>:

$$F(x,y,z) = \ln(xyz) + x^2y^2z^2 = 0$$

$$F_x = \frac{1}{x} + 2xy^2z^2 = \frac{1 + 2x^2y^2z^2}{x}$$

$$F_y = \frac{1}{y} + 2x^2yz^2 = \frac{1 + 2x^2y^2z^2}{y}$$

$$F_z = \frac{1}{z} + 2x^2y^2z = \frac{1 + 2x^2y^2z^2}{z}$$

$$\frac{\delta z}{\delta x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{1 + 2x^2y^2z^2}{x} \div \frac{1 + 2x^2y^2z^2}{z} = -\frac{z}{x}$$

$$\frac{\delta z}{\delta y} = -\frac{z}{y}$$

Exercises: Determine whether the following functions are differentiable and find the functions g_1 and g_2 as defined above.

1. a. $f(x,y) = x^2 - y^2$
   b. $f(x,y) = xy^2$
   c. $f(x,y) = y^2 + 3x + x^2$

2. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (\mathrm{x},\mathrm{y}) \neq (0,0) \\ 0 & (\mathrm{x},\mathrm{y}) = (0,0) \end{cases}$

Show that $\mathrm{f}_\mathrm{x}(0,0)$ and $\mathrm{f}_\mathrm{y}(0,0)$ exist, but f is not differentiable at $(0,0)$

3. Find $\dfrac{\delta f}{\delta x}$ and $\dfrac{\delta f}{\delta y}$ for:
   a. $\mathrm{f}(\mathrm{u},\mathrm{v}) = 2u + 3v - 4$ $\quad u = x^2 - y^2 \quad v = x^2 - 2xy + y^2$
   b. $\mathrm{f}(\mathrm{u},\mathrm{v},\mathrm{w}) = wSin(uv)$ $\quad u = \varepsilon^y \quad v = \varepsilon^{-x} \quad w = \ln(x+y)$
   c. $\mathrm{f}(\mathrm{u},\mathrm{v},\mathrm{w}) = \varepsilon^u \ln(v+w)$ $\quad u = \dfrac{1}{x+y} \quad v = xy \quad w = \varepsilon^{-y}$

4. Find the first partial derivatives of z by using implicit differentiation:
   a. $\mathrm{x}^2yz + yz^2 + x^2z^2 - 3 = 0$
   b. $\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} + 3 = 0$
   c. $Sin(xy) + Cos(xy) + Tan(xy) + xyz = 0$
   d. $x^2 + y + xz + yz^2 + 16 = 0$

Answers:

1. $a.$ Yes, $g_1 = \Delta x, g_2 = \Delta y.$ $g_1, g_2 \to 0,$ as $\Delta x, \Delta y \to 0.$

   b. Yes, $g_1 = 2y\Delta y,$ $g_2 = (x+\Delta x)\Delta y, g_{1,}g_2 \to 0, as\ \Delta x, \Delta y \to 0.$

   c. Yes, $g_1 = \Delta x, g_2 = \Delta y,$ $g_1, g_2 \to 0, as\ \Delta x, \Delta y \to 0$

2. $f_x(0,0) = 0, f_y(0,0) = 0$ so we know f_x and f_y exist, but $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist . Hence $f(x,y)$ is not differentiable at $(0,0)$

3. $a.\ \dfrac{\delta f}{\delta x} = 10x - 6y, \qquad \dfrac{\delta f}{\delta y} = 2y - 6x$

   $b.\ \dfrac{\delta f}{\delta x} = \dfrac{Sin(uv)}{x+y} - \dfrac{uwCos(uv)}{\varepsilon^x}, \quad \dfrac{\delta f}{\delta y} = vwCos(uv)\varepsilon^y + \dfrac{Sin(uv)}{x+y}$

   $c.\ \dfrac{\delta f}{\delta x} = \varepsilon^u\left[\dfrac{y}{v+w} - \dfrac{\ln(v+w)}{(x+y)^2}\right], \quad \dfrac{\delta f}{\delta y} = \varepsilon^u\left[\dfrac{x}{v+w} - \dfrac{\ln(v+w)}{(x+y)^2}\right] - \dfrac{\varepsilon^{u-y}}{v+w}$

4. $a.\ \dfrac{\delta z}{\delta x} = -\dfrac{2xyz + 2xz^2}{x^2y + 2yz + 2x^2z}, \quad \dfrac{\delta z}{\delta y} = -\dfrac{x^2z + z^2}{x^2y + 2yz + 2x^2z}$

   $b.\ \dfrac{\delta z}{\delta x} = -\dfrac{(x+y)^{-\frac{1}{2}} + (x+z)^{-\frac{1}{2}}}{(y+z)^{-\frac{1}{2}} + (x+z)^{-\frac{1}{2}}}, \quad \dfrac{\delta z}{\delta y} = -\dfrac{(x+y)^{-\frac{1}{2}} + (y+z)^{-\frac{1}{2}}}{(y+z)^{-\frac{1}{2}} + (x+z)^{-\frac{1}{2}}}$

   $c.\ \dfrac{\delta z}{\delta x} = -\dfrac{1}{x}\left[Cos(xy) - Sin(xy) + Sec^2(xy) + z\right], \dfrac{\delta z}{\delta y} = -\dfrac{1}{y}\left[Cos(xy) - Sin(xy) + Sec^2(xy) + z\right]$

   $d.\ \dfrac{\delta z}{\delta x} = -\dfrac{2x+z}{x+2yz}, \quad \dfrac{\delta z}{\delta y} = -\dfrac{2y+z^2}{x+2yz}$