Tangent Planes and Normal Lines

If f(x,y) = C and we assign a particular constant value to C, we obtain a level curve at C.

Example: If $f(x, y) = x^2 + y^2 = C$, and C = 9, we obtain $x^2 + y^2 = 9$. (A circle of radius 3). This is the level curve at C = 9. Similarly, If $f(x, y, z) = z^2 - 2x^2 - 2y^2 = C$ and C = 12 we obtain $z^2 - 2x^2 - 2y^2 = 12$. (A hyperboloid). This is the level surface a C = 12.

If we place f(x,y) = C in parametric form we get f(r(t)) = f(x(t), y(t)) = C

Where $r(t) = \langle x(t), y(t) \rangle$. The tangent vector is $r'(t) = \langle x'(t), y'(t) \rangle$.

Consider $\frac{d}{dt} f(r(t)) = \frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} C = 0$

By the chain rule $\frac{d}{dt} f = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0.$

So we have $\frac{d}{dt} f(r(t)) = \nabla f \cdot r(t) = 0.$

This shows that the gradient, ∇f, is orthogonal to the tangent $r'(t)$, that is, ∇f is normal to the level curve. Its direction vector is $\langle f_x, f_y \rangle$, the slope of the normal line is $\frac{f_y}{f_x}$, and the equation of the normal line at a point (a,b) is

$$f_y(x-a)-f_x(y-b)=0, \text{ or } x=a+f_x(a,b)t, \; y=b+f_y(a,b)t.$$

In the case of a level surface f(x,y,z) = C, at P(a,b,c), the normal line has the

equations: $$\begin{aligned} x &= a+f_x(a,b,c)t \\ y &= b+f_y(a,b,c)t \\ z &= c+f_z(a,b,c)t \end{aligned}$$

Example: Find the equations of the normal line to the surface $z=x^2+3y^2$

at the point (1,2,13). $f_x=-2x \quad f_y=-6y \quad f_z=1$

at (a,b,c)=(1,2,13)

x = 1 – 2t, y = 2 – 12t, z = 13 + t

Example: Find the equations of the normal line to the cone $z^2=x^2+y^2$ at (-3,-4,5).

$$\begin{aligned} &f(x,y,z)=z^2-x^2-y^2 \\ &\nabla f=\langle -2x,-2y,2z\rangle \\ &\text{at } (-3,-4,5),\ \nabla f=\langle 6,8,10\rangle \end{aligned}$$

x = -3 + 6t, y = -4 + 8t, z = 5 + 10t

Since the gradient will always be orthogonal to a surface curve we can always find the direction numbers for the normal as a given point. Since the normal is perpendicular to the tangent we are then able to write the equation of the tangent plane.

Example: Give the equation of the tangent plane to the surface $z = x^2 + y^2$

At the point (1,5,26). Let $f(x, y, z) = z - x^2 - y^2$. Then $\nabla f = \langle -2x, -2y, 1\rangle$. At

(1,5,26) $\nabla f = \langle -2, -10, 1\rangle$. The equation of the tangent plane is

$$-2(x - 1) - 10(y - 5) + (z - 26) = 0 \text{ or } 2x + 10y - z = 26.$$

Example: Find the point on the surface $x^2 + y^2 - 2x + 4y + z + 1 = 0$ where the tangent plane is parallel to the xy-plane.

The tangent plane will be parallel to the xy-plane only if its gradient is parallel to the z-axis, that is, it must be parallel to the unit vector $k = \langle 0,0,1\rangle$.

Now we have
$$f_x = 2x - 2,\ f_y = 2y + 4,\ f_z = 1, \text{ so that}$$
$$\nabla \mathrm{f}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \langle 2x - 2, 2y + 4, 1\rangle = \langle 0,01\rangle$$

This implies 2x – 2 = 0 and 2y + 4 = 0, or x = 1 and y = -2, so that when x = 1,

y = -2, and z = -4 there will be a horizontal tangent plane.

Example: At what point on the surface $x^2 + y^2 - 2x + 4y + z + 1 = 0$ is the tangent plane parallel to the vector $\langle 3, -1, 2\rangle$?

If the tangent plane is parallel to the vector, its normal vector must be perpendicular to the vector. If two vectors are perpendicular, their dot product must be equal to zero, that is, $N \cdot \langle 3,-1,2 \rangle$ must equal zero.

$$\nabla f = \langle 2x-2, 2y+4, 1 \rangle, \text{ so}$$
$$\langle 2x-2, 2y+4, 1 \rangle \cdot \langle 3,-1,2 \rangle = 0 \text{ implies}$$
$$3x - y = 4, \text{ so}$$
$$\langle 2,8,1 \rangle \cdot \langle 3,-1,2 \rangle = 0$$

If x = 2, y = 2, then from the original equation z = -13. At (2,2,-13) the tangent plane is parallel to the vector $\langle 3,-1,2 \rangle$.

Angle Between Two Planes:

When two planes intersect, the angle between the two planes is the same as the angle between their normals. So we can use the formula for the dot product to find the angle between the normals:

$$Cos\theta = \frac{N_1 \cdot N_2}{\|N_1\|\|N_2\|}.$$

Example: Find the angle of intersection of the following two planes:

$$3x + 6y - z - 5 = 0 \quad \text{and} \quad 9x + y + 5z - 18 = 0$$

$$N_1 = \langle 3,6,-1 \rangle;\ N_2 = \langle 9,1,5 \rangle \qquad N_1 \cdot N_2 = 28$$
$$\|N_1\| = \sqrt{46} \qquad \|N_2\| = \sqrt{107}$$
$$Cos\,\theta = \frac{28}{\sqrt{46}\sqrt{107}} = 0.3992$$
$$\theta = Cos^{-1} 0.3992$$
$$\theta = 66.5^o$$

Exercises:

1. Find the equations for the normal line and tangent plane to the following surfaces at the point given:

   $a.\ z = 5x^2 + 3y^2$ at $(1,2,17)$

   $b.\ 2x^2 + y^2 + z^2 = 2$ at $(1,0,0)$

2. Find the point(s) at which the tangent plane to the surface is horizontal.

   $a.\ z = 2x^2 + y^2$

   $b.\ 4x^2 + 9y^2 - z^2 + 36 = 0$

3. Find the angle between the two surfaces at the given point:

   $a.\ z = x^2 + y^2,\ x^2 + y^2 + z^2 = 20\ at\ (0,2,4)$

   $b.\ z = x^2 - y^2,\ z = 4\ at\ (2,0,4)$

Answers: 1.a. $x = 1 - 10t$, $y = 2 - 12t$, $z = 17 + t$; $10x + 12y - z = 17$

b. $x = 1 + 4t$; $x = 1$

2.a. (0,0,0)

b. (0,0,6)

3.a. 77.47

b. 75.96