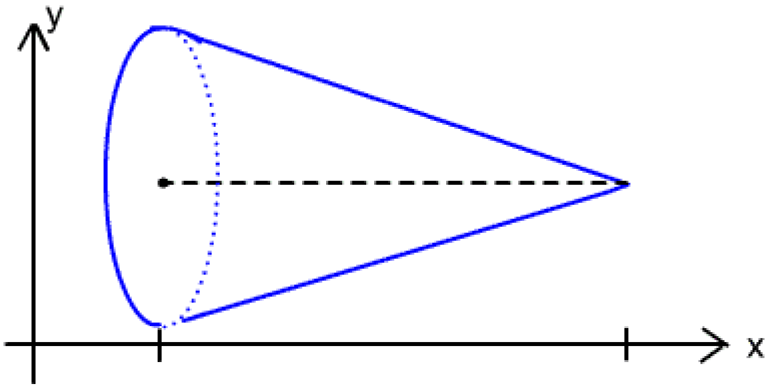
THE VOLUME OF A SOLID OF REVOLUTION

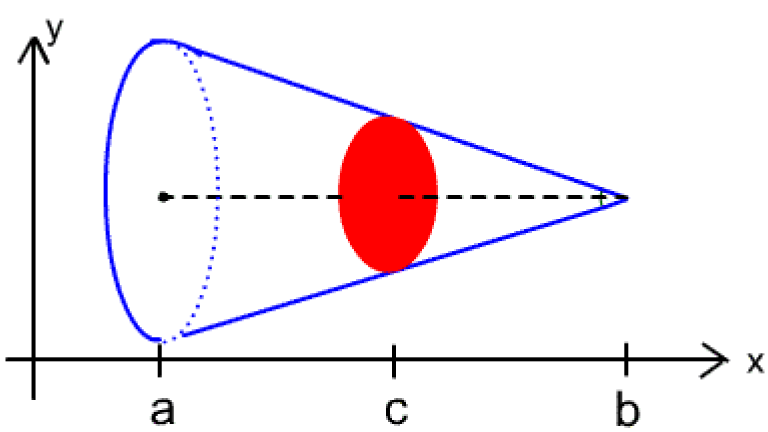
Suppose we want to find the volume of a solid. We will see that integrals can be used to do so.



Consider a solid in 3-dimensional space which is pictured above as a right circular cone.

We are interested in looking at cross-sections of the solid which are perpendicular to the x-axis.

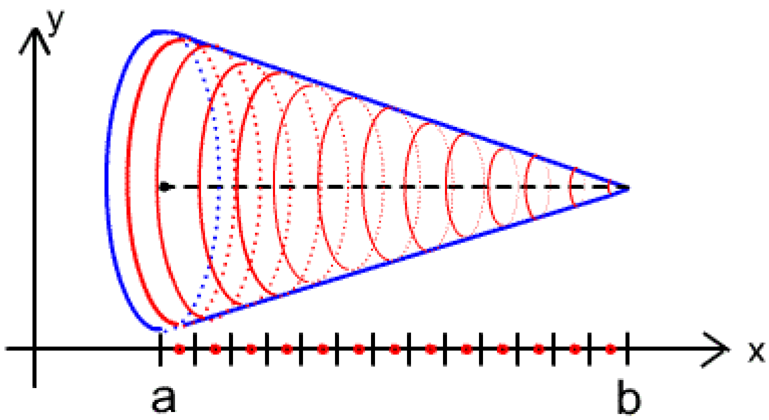
Suppose that the solid lies between x = a and x = b.



Let A(c) be the area of the cross-section at x=c.

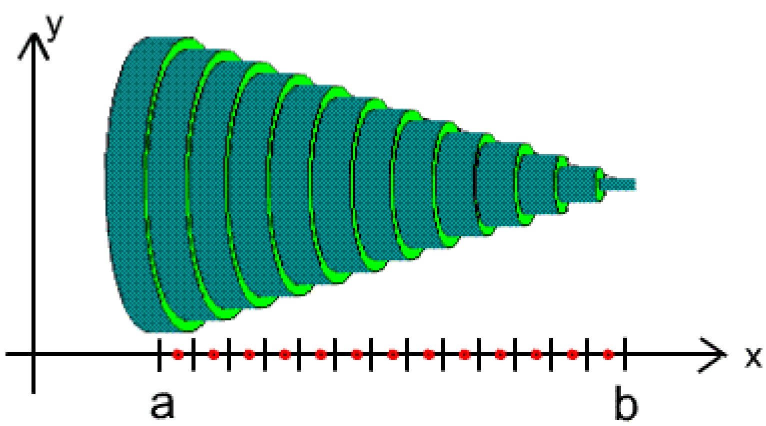
Let n be a positive integer. Subdivide [a, b] into n subintervals each of length $\Delta x = \frac{b-a}{n}$.

In each subinterval, pick x_i. Draw cross-sections at each of the x_is.

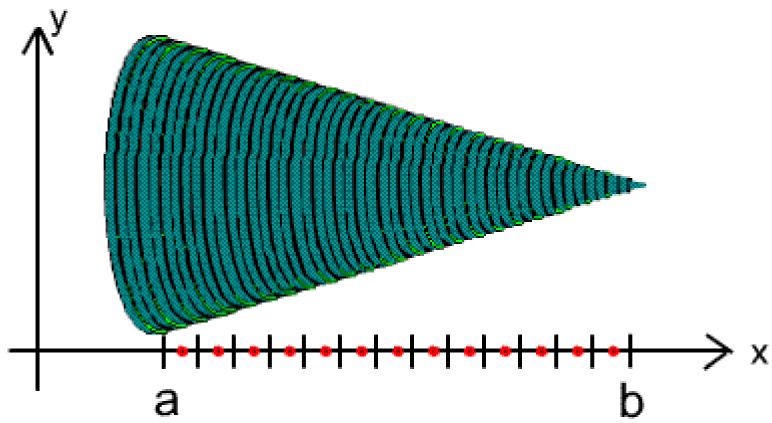


The area of the cross-section at x_i is $A(x_i)$.

Now we use the cross-sections to form n thin slices that taken together will approximate the volume of the solid.



The volume of the i^{th} disk is $A(x_i)\Delta x$. The sum of the slices, $\sum_{i=1}^{n} A(x_i)\Delta x$, gives an approximation of the volume.



If we increase n, then our approximation improves. We define the volume of the solid to be the limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\Delta x$$

if this limit exists.

If A is a continuous function, then the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\Delta x$$

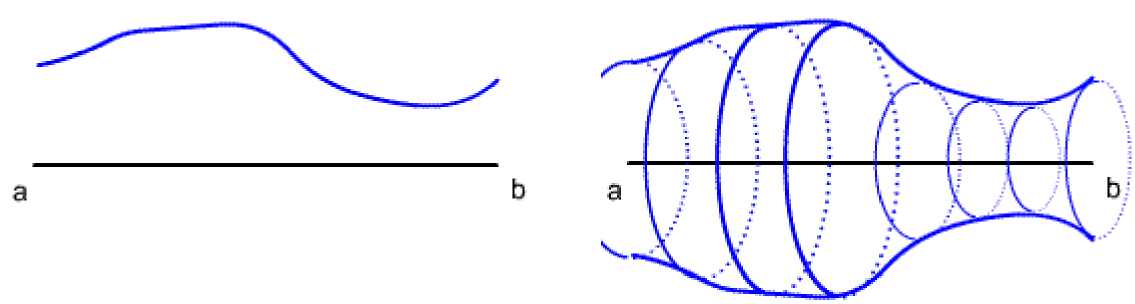
is the integral $\int_a^b A(x)dx$. Hence we define the volume of the solid to be

$$\int_a^b A(x)dx$$

where A(x) is the area of the cross-section at x.

SOLIDS OF REVOLUTION
DISK METHOD

Suppose we start with the graph of a function f(x) on [a, b], and we rotate the graph about the x-axis. The resulting solid is referred to as a solid of revolution.



If we imagine the solid consisting of thin slices, then each slice will be a circular disk with radius r = f(x) and width dx.

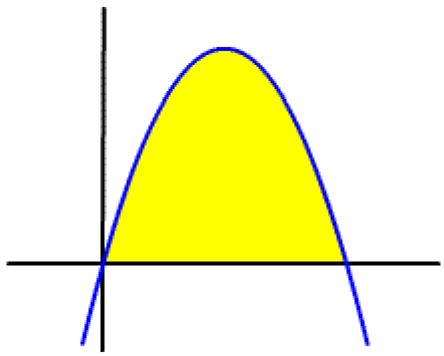
The volume of each disk will be its cross-sectional area times its width: $A(x)dx = \pi r^2 dx$, or $A(x)dx = \pi[f(x)]^2 dx$.

The volume of the solid is

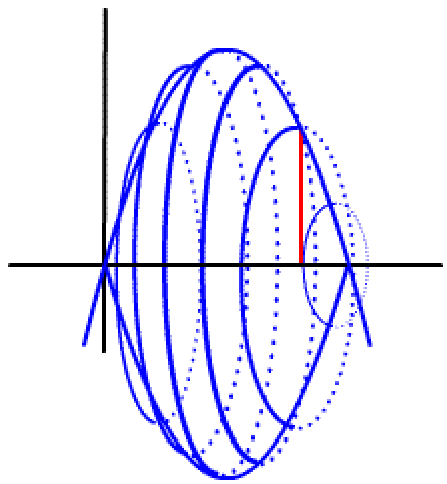
$$V = \int_a^b \pi[f(x)]^2 dx.$$

EXAMPLE

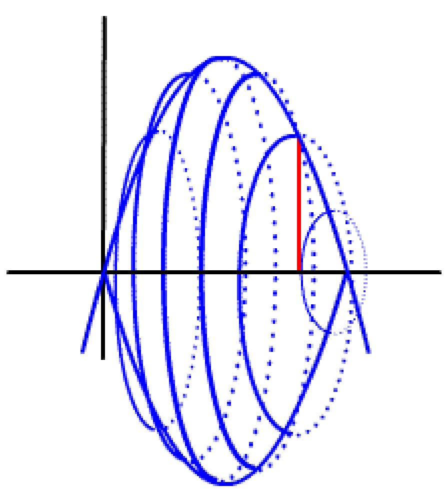
Consider the area bounded by the graph of the function $f(x) = x - x^2$ and the x-axis:



The limits of integration will be where f(x) crosses the x-axis.
$f(x) = 0$ when $x(1 - x) = 0$,
so $x = 0$ and $x = 1$.



Each disk will have cross-sectional area
$A(x) = \pi[f(x)]^2 = \pi(x - x^2)^2$,
and thickness dx.



The volume of the solid is:

$$\int_0^1 \pi (x - x^2)^2\, dx$$

$$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\, dx$$

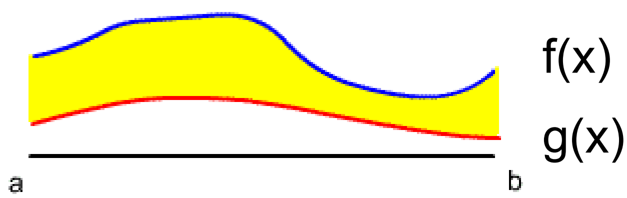
$$= \pi \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1$$

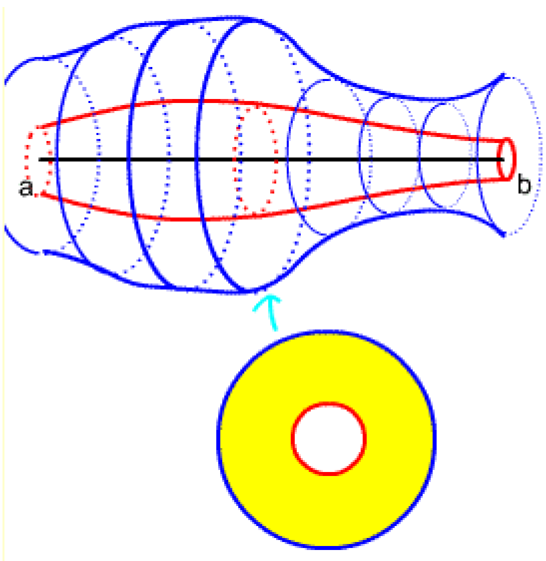
$$= \pi\left[\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right] - \pi\left[\frac{0}{3} - \frac{0}{4} + \frac{0}{5}\right]$$

$$= \pi/30$$

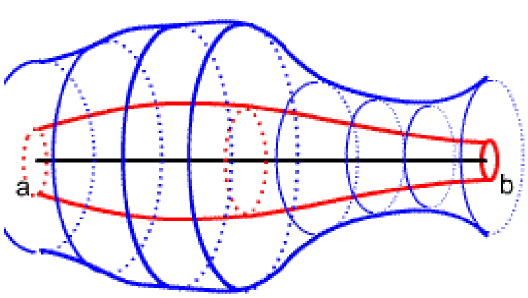
SOLIDS OF REVOLUTION
WASHERS

Suppose we start with the graphs of two functions f(x) and g(x) with $0 \le g(x) \le f(x)$ for all x in [a, b]. We want to rotate the region between these two functions about the x-axis.





Each cross-section is a circle with radius f(x) from which a circle with the same center and radius g(x) has been removed. The resulting cross-section is called a washer.



The area of a cross-section is

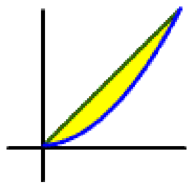
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2$$

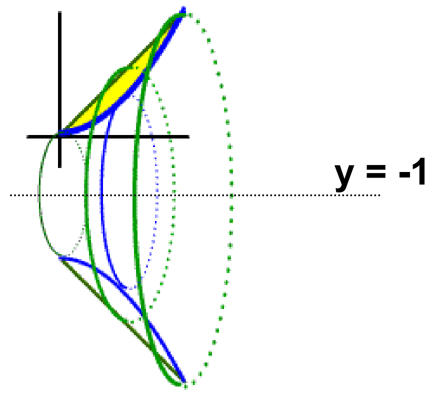
The volume of the solid is

$$V = \int_a^b \left(\pi[f(x)]^2 - \pi[g(x)]^2\right) dx$$

REVOLVING AROUND A LINE Y = C

We start with the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = x$. Notice that g(x) is the upper curve. Rotate this area about the line y = -1. We want to find the volume of the resulting solid.





The intersections of the two graphs occur when f(x) = g(x).
$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0$ and $x = 1$.
These will be the limits of integration.

Each slice will be a washer with outer radius g(x) + 1 (the distance from g(x) to the line y = -1) and inner radius f(x) + 1.

The cross-sectional area of a washer will be

$$A(x) = \pi[g(x) + 1]^2 - \pi[f(x) + 1]^2 = \pi[x + 1]^2 - \pi[x^2 + 1]^2$$
$$= \pi[2x - x^2 - x^4].$$

The volume of the solid is the integral

$$\pi \int_0^1 (2x - x^2 - x^4)dx = \pi\left[x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$
$$= \pi\left[1^2 - \frac{1^3}{3} - \frac{1^5}{5}\right] - \pi\left[0^2 - \frac{0^3}{3} - \frac{0^5}{5}\right]$$
$$= \frac{7\pi}{15}$$