Representation of Functions by Power Series

A geometric series is really the power series $a + ar + ar^2 + ar^3 + \ldots.$ whose sum

as n approaches ∞ is $\sum_0^\infty ar^n = \dfrac{a}{1-r}$ where $|r| < 1$.

Now, if $a = 1$ and $r = x$ we have the power series $1 + x + x^2 + x^3 + \ldots.$ whose

sum is $\sum_0^\infty x^n = \dfrac{1}{1-x}$ where $|x| < 1$.

Since $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$ if f(x)= $\dfrac{1}{1-x}$, we can say that the function f(x) is represented by the power series . That is,

f(x)= $1 + x + x^2 + x^3 + \ldots.$ $|x| < 1$

If f(x) looks like $\dfrac{a}{1-r}$, then f(x)= $\sum_0^\infty ar^n$ as long as $|r| < 1$.

If f(x) does not look like $\dfrac{a}{1-r}$, and we want to represent f(x) by a power series,

then we need to do some algebraic manipulation to make it look like $\dfrac{a}{1-r}$.

<u>Example</u>:

a) f(x)= $\dfrac{1}{1+x}$ Change the sign of x

f(x)= $\dfrac{1}{1-(-x)}$ $a = 1,\ r = -x$

b) f(x)= $\dfrac{1}{2-x}$ Divide numerator and denominator by 2

f(x)= $\dfrac{\frac{1}{2}}{1-\frac{x}{2}}$ $a = \dfrac{1}{2},\ r = \dfrac{x}{2}$

c) f(x)= $\frac{1}{3+4x}$ Divide numerator and denominator by 3 and change the sign of the x-term

$$f(x)=\frac{1}{3-(-4x)}=\frac{\frac{1}{3}}{1-\frac{-4x}{3}} \qquad a=\frac{1}{3},\quad r=\left(\frac{-4x}{3}\right)$$

d) f(x)= $\frac{5x}{x^2-x-6}$ Express f(x) as partial fractions, divide, and change the sign of the x-term.

$$f(x)=\frac{5x}{x^2-x-6}=\frac{3}{x-3}+\frac{2}{x+2}=\frac{-1}{1-\left(\frac{x}{3}\right)}+\frac{1}{1-\left(\frac{-x}{2}\right)}$$

$\downarrow \quad \downarrow a=1, r=\frac{-x}{2}$

$\downarrow$

$\downarrow a=-1, r=\frac{x}{3}$

e) f(x)= $\frac{1}{1-x^2}$ $a=1, r=x^2$ so that f(x) = $\sum_0^\infty x^{2n}$

The power series $\sum_0^\infty ar^n$ converges when $|r|<1$ or when -1 < x < 1. x is centered at zero. If a power series is centered at c, then there is a number R (called the radius of convergence) such that the series converges absolutely for $|x-c|<R$ and diverges for $|x-c|>R.$ The Ratio Test usually gives us R.

For the Geometric power series R=1 so $|x-c|<1$ or -1 < x < 1. Suppose the power series is centered at c = 2. Then -1 < x – 2 < 1 implies that 1 < x < 3.

The set of all values of x for which the power series converges is called the interval of convergence. The end points are tested to see whether the interval is open, closed, or half-open, half-closed.

The power series for the function f(x) = $\frac{1}{1-x} = \sum_{0}^{\infty} x^n$ is centered at c=0, that is

-1 < x < 1.

Suppose we wanted this power series to be centered at c = 2, that is, $|x-2|$ < 1.

We take the denominator (1 – x), replace the x with (x – 2), and add some number A so that the denominator doesn't really change. In this case

(1 – x) = A – (x – 2) = (A + 2) – x $\Rightarrow$ A + 2 = 1

A, therefore, would have to equal -1.

So (1 – x) = -1 – (x – 2)

Then f(x) = $\frac{1}{1-x} = \frac{1}{-1-(x-2)} = \frac{-1}{1-(2-x)}$

Now a = -1 , r = 2 – x and

f(x) = $\sum_{0}^{\infty} -(2-x)^n$ $\quad |2-x| < 1$

$|2-x| < 1 \quad \Rightarrow 1 < x < 3$

The power series is centered at c = 2 and its interval of convergence is (1,3).

Represent the following functions by power series:

(1) $f(x)=\dfrac{1}{(1-x)^2}$

(2) $f(x)=\dfrac{6}{3-2x}$

(3) $f(x)=\dfrac{1}{x^2-x-2}$

(4) $f(x)=\dfrac{1}{2x+2}$

Answers:

(1) $\sum_{1}^{n} nx^{n-1}$ (2) $\sum_{0}^{n} 2^{n+1}\left(\frac{x}{3}\right)^n$ (3) $\frac{1}{3}\sum_{0}^{n}\left[(-1)^{n+1}-\left(\frac{1}{2}\right)^{n+1}\right]x^n$ (4) $\sum_{0}^{n}\frac{1}{2}(-x)^n$