PARTIAL DIFFERENTIATION
(MAT 204)

To partially differentiate a function of several variables we hold all the other variables as constants while we differentiate with respect to the selected variable. So if z = f(x, y) and we want the partial derivative of z with respect to x, we would regard y as a constant and differentiate z with repect to x, that is, $\frac{\delta z}{\delta x}$. Different notations are used to show this partial derivative:

$$f_x \, , \frac{\delta f(x,y)}{\delta x}, \, z_x \, , \frac{\delta z}{\delta x}$$

Examples:

$f(x,y) = x + y^2$, Find f_x and f_y

$$f_x = \frac{\delta}{\delta x}\left(x + y^2\right) = 1 + 0 = 1 \quad \left(\text{y is considered a constant}\right)$$

$$f_y = \frac{\delta}{\delta y}\left(x + y^2\right) = 0 + 2y = 2y \quad \left(\text{x is considered a constant}\right)$$

$z = f(x,y) = xy^2$

$$\frac{\delta z}{\delta x} = \frac{\delta}{\delta x}\left(xy^2\right) = y^2 \quad \text{(y is considered a constant)}$$

$$\frac{\delta z}{\delta y} = \frac{\delta}{\delta y}\left(xy^2\right) = 2xy \quad \text{(x is considered a constant)}$$

$f(x,y) = \frac{x}{y}$

$$f_x = \frac{\delta}{\delta x}\left(\frac{x}{y}\right) = \frac{1}{y} \quad \text{(y is considered a constant)}$$

$$f_y = \frac{\delta}{\delta y}\left(\frac{x}{y}\right) = -\frac{x}{y^2} \quad \text{(x is considered a constant)}$$

What does a partial derivative mean? If you have a surface z = f(x, y), the partial derivative gives the slope of the surface in the x-direction and the y-direction at a given point (x_0, y_0, z_0).

Example: What is the slope of the surface

$$z = f(x,y) = 3x - x^2y^2 + 2x^3y \text{ at the point } (1,2,3)?$$

$$f_x = 3 - 2xy^2 + 6x^2y \text{ ; } f_y = -2x^2y + 2x^3$$
$$\text{at } (1,2) \text{ we have } f_x = 7 \;\; f_y = -2$$

The slope of the surface at the point (1, 2, 3) in the x-direction is 7, while the slope in the y-direction is -2.

Higher-Order Partial Derivatives:

If $z = f(x, y)$ and f_x and f_y are the first order partial derivatives, then the second order partial derivatives are $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ where

$$f_{xx} = \frac{\delta^2 f}{\delta x^2}; \;\; f_{yy} = \frac{\delta^2 f}{\delta y^2}; \;\; f_{xy} = \frac{\delta^2 f}{\delta y \delta x}; \;\; f_{yx} = \frac{\delta^2 f}{\delta x \delta y} \text{ and}$$

$$\frac{\delta^2 f}{\delta x^2} = \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right); \;\; \frac{\delta^2 f}{\delta y \delta x} = \frac{\delta}{\delta y}\left(\frac{\delta f}{\delta x}\right)$$

Notice that f_{xy} means that we get f_x first and then we differentiate with respect to y.

Example: Find all first and second order derivatives if

$$f(x,y) = 4x^3 - 3xy^2 + y\varepsilon^x$$

$$f_x = 12x^2 - 3y^2 + y\varepsilon^x$$
$$f_y = -6yx + \varepsilon^x$$
$$f_{xx} = 24x + y\varepsilon^x \qquad f_{yy} = -6x$$
$$f_{xy} = -6y + \varepsilon^x \qquad f_{yx} = -6y + \varepsilon^x$$

Note that if $f_x, f_y, f_{xy},$ and f_{yx} are all continuous, then $f_{xy} = f_{yx}$.

Exercises: Find all first and second order derivatives

1. $f(x,y) = \sin(3x + 2y)$
2. $f(x,y) = \varepsilon^{x^2+y}$
3. $f(x,y) = x\ln(x^2 + y^2)$
4. $f(x,y) = \varepsilon^x \cos y$
5. $f(x,y,z) = \varepsilon^{xyz}$

Answers:

1. $f_x = 3\cos(3x + 2y) \qquad f_y = 2\cos(3x + 2y)$

   $f_{xx} = -9\sin(3x + 2y) \qquad f_{yy} = -4\sin(3x + 2y)$

   $f_{xy} = f_{yx} = -6\sin(3x + 2y)$

2. $f_x = 2x\varepsilon^{x^2+y} \qquad f_y = \varepsilon^{x^2+y}$

   $f_{xx} = 2(2x^2 + 1)\varepsilon^{x^2+y} \qquad f_{yy} = \varepsilon^{x^2+y}$

   $f_{xy} = f_{yx} = 2x\varepsilon^{x^2+y}$

3. $f_x = \ln\left(x^2 + y^2\right) + \dfrac{2x^2}{x^2 + y^2}$

$f_y = \dfrac{2xy}{x^2 + y^2}$

$f_{xy} = \dfrac{2y^3 - 2x^2 y}{\left(x^2 + y^2\right)^2} = f_{yx}$

$f_{xx} = \dfrac{2x^3 + 6xy^2}{\left(x^2 + y^2\right)^2}$

$f_{yy} = \dfrac{2x^3 - 2xy^2}{\left(x^2 + y^2\right)^2}$

4. $f_x = \varepsilon^x \cos y \qquad f_y = -\varepsilon^x \sin y$

$f_{xx} = \varepsilon^x \cos y \qquad f_{yy} = -\varepsilon^x \cos y$

$f_{xy} = f_{yx} = -\varepsilon^x \sin y$

5. $f_x = yz\varepsilon^{xyz} \qquad f_y = xz\varepsilon^{xyz} \qquad f_z = xy\varepsilon^{xyz}$

$f_{xx} = y^2 z^2 \varepsilon^{xyz} \quad f_{yy} = x^2 z^2 \varepsilon^{xyz} \quad f_{zz} = x^2 y^2 \varepsilon^{xyz}$

$f_{xy} = \left(xyz^2 + z\right)\varepsilon^{xyz} \quad f_{xz} = \left(xy^2 z + y\right)\varepsilon^{xyz} \quad f_{yz} = \left(x^2 yz + x\right)\varepsilon^{xyz}$