Parametric Equations

Sometimes an equation in two variables, say x and y, can be changed into two equations where the variable x and y are expressed in terms of a third variable, say t. These new equations are called parametric equations and the parameter is t. For example, suppose $y = 5x - 2$. If we let $x = t$, then $y = 5t - 2$. We could have said, "Let $x = (t + 2/5)$", then y would equal 5t. This shows that an equation can have more than one parametric representation, or that the parametric representation of an equation is not unique.

To graph an equation in parametric form we do exactly as we did with equations in two variables: we give values to the parameter and compute the corresponding values of x and y.

For example: $x = t \quad y = 5t - 2$. We set up a table as follows:

t =	0	1	2	3
x =	0	1	2	3
y =	-2	3	8	13

Then we graph the values of x and y to see what the curve looks like. If on the curve for each (x, y) we put the corresponding value of t, and then we see in what direction the curve is going as t increases, we are talking about the orientation of the curve. It could be from right to left, counterclockwise, clockwise, up and down, etc. We have to be careful and make sure that any restrictions on the domain of t are transferred to those of x and y, that is, if a value of t is not in the domain of t, we can't use that value to compute x or y.

If $x = \frac{1}{\sqrt{(2-t)}}$, we see that t can never be greater than 2 nor can it equal 2.

Sometimes we are asked to eliminate the parameter in order to get back to the equation in x and y only. Depending on the parametric equations this could be easy or difficult.

<u>Example</u>: Eliminate the parameter from x = 3t – 1 and y = 2t + 1

From the first equation we get t = (x + 1)/3. Substitute this value for t in the second equation and we get y = 2(x + 1)/3 + 1 or 3y = 2x + 5.

<u>Example</u>: Suppose x = 4 + 2cos t and y = -1 + 2sin t and we need to eliminate the parameter t. From the first equation we get cos t = (x – 4)/2 and from the second, sin t = (y + 1)/2. We now make use of the trigonometric identity

$(\sin t)^2 + (\cos t)^2 = 1$ and write $\frac{(x-4)^2}{4} + \frac{(y+1)^2}{4} = 1$, which reduces to the equation

$(x-4)^2 + (y+1)^2 = 4$ which we recognize as the equation of a circle whose center is at (4, -1) and whose radius is equal to 2.

Some operations and formulas in calculus are based on equations of the form y = f(x). When y and x are given as parametric equations the operations and formulas change. Take derivatives for example:

<u>For first derivatives</u> $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

<u>Example</u>: $x = t^2 \quad y = t^3 - 4t$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2 - 4$$

$$\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx} = \frac{(3t^2 - 4)}{2t} = \frac{3t}{2} - \frac{2}{t}$$

<u>For second derivatives</u> $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2} + \frac{2}{t^2}}{2t} = \frac{3}{4t} + \frac{1}{t^3}$

<u>For third derivatives</u> $\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)}{\frac{dx}{dt}} = \frac{\frac{-3}{4t^2} - \frac{3}{t^4}}{2t} = \frac{-3}{8t^3} - \frac{3}{2t^5}$

Other formulas which change when x and y are given as parametric equations are

<u>Arc Length</u>

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Area of Surface of Revolution

x = f(t)   y = g(t) on $a \le t \le b$

Revolution about the x-axis $g(t) \ge 0$

$$S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Revolution about the y-axis $f(t) \ge 0$

$$S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example of Arc Length:

$$x = e^{-t}\cos(t) \qquad y = e^{-t}\sin(t) \qquad 0 \le t \le \frac{\pi}{2}$$

$$\frac{dx}{dt} = -e^{-t}\cos(t) - e^{-t}\sin(t) = -e^{-t}\left(\cos(t) + \sin(t)\right)$$
$$\frac{dy}{dt} = -e^{-t}\sin(t) + e^{-t}\cos(t) = e^{-t}\left(\cos(t) - \sin(t)\right)$$

$$\left(\frac{dx}{dt}\right)^2 = e^{-2t}\left(\cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)\right) = e^{-2t}\left(1 + 2\cos(t)\sin(t)\right)$$
$$\left(\frac{dy}{dt}\right)^2 = e^{-2t}\left(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)\right) = e^{-2t}\left(1 - 2\cos(t)\sin(t)\right)$$
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{-2t}$$
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^{-t}$$

$$S = \int_0^{\frac{\pi}{2}} \sqrt{2}e^{-t}\,dt = -\sqrt{2}e^{-t}\Big]_0^{\frac{\pi}{2}} = -\sqrt{2}\left\{e^{\frac{-\pi}{2}} - e^0\right\} = \sqrt{2}\left(1 - e^{\frac{-\pi}{2}}\right)$$

Example of Area of Surface of Revolution:

$$x = a\cos^3\theta \qquad y = a\sin^3\theta \qquad 0 \le \theta \le \pi$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta\sin\theta \qquad\qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\cos^2\theta\sin^2\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 3a\cos\theta\sin\theta$$

Note that this square root must be positive. The limits of integration are from zero to π. In the second quadrant the cosine is negative. Since the curve is symmetric with respect to the y-axis, we can integrate between zero and π/2, and then multiply by 2.

$$S = 2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta \cdot 3a\cos\theta\sin\theta \cdot d\theta$$

$$= 6\pi a^2\int_0^{\frac{\pi}{2}} \sin^4\theta \cdot \cos\theta \cdot d\theta = \frac{6\pi a^2}{5}\sin^5\theta\Big]_0^{\frac{\pi}{2}} = \frac{6\pi a^2}{5}$$

Multiplying by 2 we get the area to be $\frac{12\pi a^2}{5}$.

Exercises:

(1) a, b are positive numbers. Given the parametric equations

$x = a\cos t, \ y = b\sin t$

What is the curve in rectangular coordinates?

(2) Find the area of the surface generated by revolving the curve given by

$x = t^3$ and $y = t + 2, \ 1 \le t \le 2$ about the y-axis.

(3) Find the arc length of the curve given by $x = t^2$ and $y = 4t^3 - 1$ on the

interval $-1 \le t \le 1$

(4) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

a. $x = 2t^3 - 1 \qquad y = t^2 + 2$

b. $x = 2\sin\theta \qquad y = 1 + \tan\theta$

c. $x = \theta - \sin^2\theta \qquad y = \theta - \cos^2\theta$

(5) Eliminate the parameter t

$$x = t^2 + 3t \qquad y = t + 1$$

Answers:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (2) $\frac{\pi}{27}(145\sqrt{145} - 10\sqrt{10})$ (3) $\frac{1}{27}\left[37^{\frac{3}{2}} - 1\right]$

(4) a. $\frac{1}{3t}; \ \frac{-1}{18t^4}$ b. $\frac{\sec^3\theta}{2}; \ \left(\frac{3}{4}\right)\sec^4\theta\tan\theta$ c. $\frac{1+\sin 2\theta}{1-\sin 2\theta}; \ \frac{4\cos 2\theta}{(1-\sin 2\theta)^3}$

(5) $x = y^2 + y - 2$