LINES IN SPACE

We usually describe lines in the plane by specifying a point and the slope of the line. In space we use the properties of vectors to describe a line. If L is any line in space and we take any two points on the line, say (x_1, y_1, z_1) and (x_2, y_2, z_2), and subtract the coordinates, we get what is called a direction or position vector $v = \langle (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) \rangle = \langle a, b, c \rangle$. We call the numbers a, b, c the direction numbers of the line.

Example: A line L has the following two points on it: (2, 3, 1), (4, 8, 3). What is L's direction vector and what are its direction numbers?

The direction vector is $\langle (4-2), (8-3), (3-1) \rangle = \langle 2,5,2 \rangle$. The direction numbers are 2,5,2.

If we take the direction vector $\langle a, b, c \rangle$ and turn it into a unit vector by dividing by its magnitude, that is, by $\sqrt{a^2 + b^2 + c^2}$, then the numbers

$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}$, and $\frac{c}{\sqrt{a^2+b^2+c^2}}$ are called direction cosines.

Example: In the previous example, what are the direction cosines of the direction vector $\langle 2,5,2 \rangle$?

$\sqrt{2^2 + 5^2 + 2^2} = \sqrt{33}$, so the direction cosines are $\frac{2}{\sqrt{33}}, \frac{5}{\sqrt{33}}, \frac{2}{\sqrt{33}}$.

When a unit direction vector is moved so that its tail is at the origin (standard position vector), we call the angles that the vector makes with the coordinate axes its direction angles.

Example: What are the direction angles for the unit position vector $\frac{1}{\sqrt{33}}\langle 2,5,2\rangle$?

$$\text{Cos}\,\alpha = \frac{2}{\sqrt{33}} = 0.3482 \quad \alpha = Cos^{-1}(0.3482) = 69.6^0$$

$$\text{Cos}\,\beta = \frac{5}{\sqrt{33}} = 0.8704 \quad \beta = Cos^{-1}(0.8704) = 29.5^0$$

$$\text{Cos}\,\gamma = \frac{2}{\sqrt{33}} = 0.3482 \quad \gamma = Cos^{-1}(0.3482) = 69.6^0$$

Note that $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = 1$.

Equations of Lines in Space:

The line L starting at a point (x_o, y_o, z_o) in space is evidently parallel to its position vector $\langle a,b,c\rangle$. The line is therefore a multiple of the vector $\langle a,b,c\rangle$, say t$\langle a,b,c\rangle$.

The line then is $\langle x_0, y_0, z_0\rangle$ plus t$\langle a,b,c\rangle$ or $\langle x_0 + at, y_0 + bt, z_0 + ct\rangle$.

These are the parametric equations of the line:

$$x = x_o + at \qquad y = y_0 + bt \qquad z = z_o + ct$$

Example: What are the parametric equations of the line through the point (3, -2, 4) with a position vector equal to $\langle 2,3,4 \rangle$ (parallel to $\langle 2,3,4 \rangle$)?

x = 3 + 2t y = -2 + 3t z = 4 + 4t

Example: Does the point (-1, -5, -1) lie on the line above?

The point (-1, -5, -1) would have to satisfy the parametric equations of the line to lie on the line, that is -1 = 3 + 2t, -5 = -2 + 3t, -1 = 4 + 4t. Solving the first two equations for t gives t = -1. Substituting t = -1 in the third equation gives x = 0. Since –1 does not equal zero, the point (-1, -5, -1) does not lie on the line.

If we take parametric equations for the line and solve each for the parameter t and equate results we obtain:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are called the symmetric equations of the line. Note that the denominators are the direction numbers. The symmetric equations for the parametric equations above are: $\frac{x-3}{2} = \frac{y+3}{3} = \frac{z-4}{4}$.

Angles Between Lines:

The angle between two lines is defined to be the angle between their direction vectors, even if the two lines are not in the same plane. Let v_1 and v_2 be the direction vectors of two lines. Then from the dot product of two vectors we have:

$$Cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|}$$ where θ is the angle between the two vectors.

<u>Example</u>: Find the angle between the two lines x = 4t, y = 1 – 3t, z = 2 + t and x = 2 – 3t, y = 1 – 4t, z = 7.

The direction vectors are $v_1 = \langle 4,3,1 \rangle$, $v_2 = \langle -3,-4,0 \rangle$

$$\|v_1\| = \sqrt{16+9+1} = \sqrt{26}$$

$$\|v_2\| = \sqrt{9+16+0} = 5$$

$$v_1 \bullet v_2 = -12 + 12 + 0 = 0$$

$$Cos\theta = \frac{0}{5\sqrt{26}} = 0$$

$$\theta = \frac{\pi}{2}$$

Exercises:

1. Find the direction vector for each line

   a. x = 1 + 2t, y = 3, z = 3 + 3t

   b. The line through (2, 3, 4) and (5, 5, 1)

2. Find the angle between the two lines x = t, y = t, z = -t and x = 2 + 2t,

   y = 3 + 3t, z = -1.

3. Find the parametric equations for the lines

   a. Through (2, -1, 3) and parallel to the x-axis.

   b. Through (2, -1, 3) and parallel to the z-axis

   c. Through (1, 5, -1) and the origin

4. Find the direction cosines for $\frac{x-1}{-2} = \frac{y+3}{3} = \frac{z}{-5}$

5. Change the equation of the line in Exercise 4 to the parametric form.

Answers:

1a. $\langle 2,0,3 \rangle$ 1b. $\langle 3,2,-3 \rangle$

2. 36.8 degrees

3a. x = 2 + t, y = -1, z = 3 3b. x = 2, y = -1, z = 3t

3c. x = t, y = 5t, z = -t

4. $\frac{-2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{-5}{38}$ 5. x = 1 – 2t, y = -3 + 3t, z = -5t