LIMITS (FUNCTIONS OF TWO VARIABLES)

One of the main differences between limits of functions of one variable and limits of functions of two variables is that limits of functions of one variable at a point x = a are considered in an interval on the number line while limits of functions of two variables at a point x = a, y = b are considered in a disc in the xy-plane. For example, with a function of one variable at x_0, $|x - x_0| < \delta$, this would mean that within the interval the distance of x from x_0 is always less than δ. With a function of two variables at (x_0, y_0), $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, this would mean that the point (x_0, y_0) lies within a circle whose radius is δ.

Another main difference is that to find the limit of a function of one variable at a point x = a, we approached a from both the left side and the right side. If both limits were the same, the function had a limit. To find the limit of a function of two variables at (x_0, y_0) we must show that the limit is the same no matter from which direction we approach (x_0, y_0).

IMPORTANT! If we approach (x_0, y_0) from two different directions and get two different results, f(x,y) does not have a limit. If we approach (x_0, y_0) from two different directions and get the same result, it doesn't necessarily mean f(x,y) has that limit. We have to get the same limit no matter from which direction we approach (x_0, y_0). To do this we would sometimes have to use the definition of the limit of a function of two variables in order to be sure we have the right limit.

Definition: f(x,y) is said to have the limit L as $(x,y) \to (x_0, y_0)$ provided that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

We will use this definition in some of the examples below.

<u>Example</u>: Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2 + y^2}$

Let x = 0 first. $\dfrac{x^2}{x^2 + y^2} = \dfrac{0}{0 + y^2} = 0$

Now let y = 0 first. $\dfrac{x^2}{x^2 + 0} = \dfrac{x^2}{x^2} = 1$

Since we got two different results, the limit does not exist.

Example: Find the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

Let x = 0 first. $\dfrac{-y^2}{y^2} = -1$

Next, let y = 0 first. $\dfrac{x^2}{x^2} = 1$

Again, the limit does not exist.

Example: Find the limit $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$

Let $(x,y) \to 0$ along the line y = mx. Then

$$\frac{xy}{x^2 + y^2} = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1+m^2}$$

This shows that the limit depends on the choice of m. Therefore, the limit does not exist.

When we use the definition of a limit to show that a particular limit exists, we usually employ certain key or basic inequalities such as:

$$|x| < \sqrt{x^2 + y^2} \qquad |y| < \sqrt{x^2 + y^2}$$

$$\frac{x}{x+1} < 1 \qquad \frac{x^2}{x^2 + y^2} < 1$$

$$|x - a| = \sqrt{(x-a)^2} \le \sqrt{(x-a)^2 + (y-a)^2}$$

<u>Example</u>: Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$

Let x = 0 $\quad \frac{x^2 y}{x^2+y^2} = 0$

Let y = 0 $\quad \frac{x^2 y}{x^2+y^2} = 0$

We suspect that the limit might be zero. Let's try the definition with L=0.

$$\left|f(x,y)-L\right| < \varepsilon \quad \text{if } \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

$$\left|f(x,y)\right| = \left|\frac{x^2 y}{x^2+y^2}\right| = |y|\left|\frac{x^2}{x^2+y^2}\right|$$

Now, since $\left|\frac{x^2}{x^2+y^2}\right| < 1$ then $|y|\left|\frac{x^2}{x^2+y^2}\right| < |y|$

So we then have $|y| < \sqrt{y^2+x^2} = \sqrt{(y-0)^2+(x-0)^2} < \delta$

Therefore, if $\delta = \varepsilon$, the definition shows the limit does equal zero.

Exercises: Determine whether the following limits exist.

1. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

2. $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

3. $\lim_{(x,y)\to(0,0)} \frac{x^2y^3+x^3y^2-5}{2-xy}$

4. $\lim_{(x,y)\to(\pi,\pi)} x\sin\left(\frac{x+y}{4}\right)$

5. $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$

6. $\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+y^2}$

7. $\lim_{(x,y)\to(0,0)} \frac{xy-y^2}{x^2+y^2}$

Answers: 1. No 2. Yes 3. -5/2

4. π 5. No 6. No 7. No