LAGRANGE MULTIPLIERS

Many problems involving extrema of functions of two or more variables also pose a condition or constraint on the values that the variables may take on while reaching an extrema. For example, finding the maximum area of a rectangle with a given or fixed perimeter, or finding the maximum product of three numbers whose sum is fixed. The method of Lagrange is based on gradients to both the function and the constraint being parallel, that is, one gradient is a scalar multiple of the other. The method works as follows:

Suppose we want to maximize f(x,y) = 0 subject to g(x,y) = 0.

We set up three equations:
$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = 0$$

The variable λ is called the Lagrange multiplier. We generally solve the first equation for λ and substitute the solution in the second equation. We then solve the second equation for x or y, and substitute it in the third.

Example: Find the maximum area of a rectangle whose perimeter is equals k.

Maximize A = f(x,y) = xy, subject to P = g(x,y) = 2x + 2y - k.

Setting $f_x(x,y) = \lambda g_x(x,y) \ and \ f_y(x,y) = \lambda g_y(x,y)$ gives

$$y = 2\lambda \ and \ x = 2\lambda$$

Substituting in the constraint gives x = y = k/4

Example: Find the dimensions of the rectangular solid of maximum volume whose surface area is fixed.

Maximize V = f(x,y,z) = xyz, subject to S = g(x,y,z) = 2yz + 2xy + 2xz - k

Set $$\begin{aligned} f(x,y,z) &= \lambda g(x,y,z) \\ xyz &= \lambda(2yz + 2xy + 2xz - k) \end{aligned}$$

Taking partial derivatives gives,

$yz = 2(y + z)\lambda$

$xz = 2(x + z)\lambda$

$xy = 2(x + y)\lambda$

Solving the first equation, $\lambda = \dfrac{yz}{2(y+z)}$ and substituting in the second and third equation, we have

$$xz = \frac{2(x+z)(yz)}{2(y+z)} \Rightarrow x = y$$
$$xy = \frac{2(x+y)(yz)}{2(y+z)} \Rightarrow y = z$$

Substituting these values in the constraint 2yz + 2xy + 2xz = k gives

$6y^2 = k$, so that

$$x = y = z = \sqrt{\frac{k}{6}}.$$

Example: The temperature at a point (x, y, z) on the surface of a sphere $x^2+y^2+z^2=4$ is given by $T(x,y,z)=x^2yz$. Find the points of maximum temperature.

Maximize $f(x,y,z)=x^2yz$ subject to $g(x,y,z)=x^2+y^2+z^2-4$

Taking partial derivatives gives

$$x(yz-\lambda)=0$$
$$x^2z-2y\lambda=0$$
$$x^2y-2z\lambda=0$$

From the first equation x = 0 or λ = yz. If x = 0, T = 0 so this is not a maximum. If λ = yz, then substituting in the second equation gives $x^2z-2y^2z=0$ from which $x=\pm\sqrt{2}y$.

From the third equation we get $x^2y-2z^2y=0$, so $x=\pm\sqrt{2}z$.

Substituting in the constraint $x^2+y^2+z^2=4$, we get $x=\pm\sqrt{2}$, from which $y=\pm1$, $z=\pm1$. Points of maximum temperature are $(\pm\sqrt{2}, \pm1, \pm1)$.

Example: What is the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$?

Maximize A = f(x,y) = xy (first quadrant)

Subject to $g(x,y)=\frac{x^2}{a_2}+\frac{y^2}{b^2}-1$

Taking partial derivatives gives

$$y=\frac{2x\lambda}{a^2} \; or \; \lambda=\frac{a^2y}{2x}$$
$$x=\frac{2y\lambda}{b^2} \; or \; \lambda=\frac{b^2x}{2y}$$

So, $\frac{a^2y}{2x}=\frac{b^2x}{2y} or\; a^2y^2=b^2x^2$. Substituting in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ yields

$x=\frac{a\sqrt{2}}{2} \; and \; y=\frac{b\sqrt{2}}{2}$.

The area A = 4xy = $4(\frac{a\sqrt{2}}{2})(\frac{b\sqrt{2}}{2})=2ab.$

Sometimes we are asked to maximize a function subject to more than one constraint. In this case we use more than one Lagrange multiplier.

Example:

Maximize f(x,y,z) = xy + yz subject to g(x,y) = x + 2y – 5, and h(x,z) = x – 4z.

Set F(x,y,z) = xy + yz - λ (x+2y-5)- γ (x-4z)

Taking partial derivatives and setting them equal to zero,

$$F_x = y - \lambda - \gamma$$
$$F_y = x + z - 2\lambda$$
$$F_z = y + 4\gamma$$

Solving the first and third equations gives $\gamma = \frac{-y}{4}, \lambda = \frac{5y}{4}$

Substituting for λ in the second equation and multiplying by 4 yields

. 4x + 4z – 10y = 0, and the second constraint is x – 4z = 0. From which we find x = 2y, and y = 2z. Substituting in the first constraint yields z = 5/8, and x = 5/2, y = 5/4. The critical points are (5/2, 5/4, 5/8) and the maximum value of F is 125/32.

Problems:

1. Find the closest point on the surface z = 2x + 3y + 5 to the origin.
2. The bottom of a box costs 4 times as much per square inch as do the sides (no top). For a given volume k, find the dimensions of the cheapest box.
3. Find the extrema of $f(x, y) = x^2 + 5xy + y^2$ subject to $x^2 + y \le 1$.
4. Temperature is given by $2x^2 + y^2 - y$. Find the hottest and coldest points on $x^2 + y^2 \le 1$.

Answers: 1. (-5/4, -15/8, 5/8)

2. $\sqrt[3]{\frac{k}{2}} \times \sqrt[3]{\frac{k}{2}} \times 2\sqrt[3]{\frac{k}{2}}$

3. Max at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, Min at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$. Saddle point at (0, 0).

4. Hottest points at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, coldest point at (0, 1/2).