L'Hôpital's Rule

Finding the limit of the quotient of two functions [f(x)/g(x)] as the independent variable approaches a certain value *c* can be difficult if direct substitution of the value *c* produces an indeterminate form.

Indeterminant forms are: **1st group**: $\frac{0}{0}$ $\frac{\infty}{\infty}$ $\frac{-\infty}{\infty}$ $\frac{\infty}{-\infty}$ $\frac{-\infty}{-\infty}$

2nd group: $0\bullet\infty$ $0\bullet-\infty$ 1^{∞} 0^0 ∞^0 $\infty-\infty$

Indeterminant forms do not guarantee a limit exists, or what it may be if it did exist.

Some forms that look similar are actually determinant:

		limit
$\infty+\infty$	$\rightarrow$	∞
$-\infty-\infty$	$\rightarrow$	$-\infty$
0^{∞}	$\rightarrow$	0
$0^{-\infty}$	$\rightarrow$	∞

If you find one of these forms on initial substitution, you're done! (For example, substitution of zero in $\lim_{x\to 0^+} x^{1/x}$ yields $0^{\infty}=0$.)

L'Hôpital's Rule states that if the limit of f(x)/g(x) as x approaches *c* produces one of the indeterminant forms in the 1st group above, then you can take the derivatives of the top and the bottom functions (separately) and see if that fixes the problem. In other words:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

You can use the rule multiple times, provided you start with one of the 1st group forms each time. If you get an indeterminant form of the 2nd group, you need to manipulate it to get a 1st group form. So check the form you have first (before taking the derivatives) – make sure it's legal.

For example: $\lim_{x\to 0}\frac{e^x-1}{x}$ produces the indeterminant form $\frac{0}{0}$ and the rule applies,

but rewriting the limit as $\lim_{x\to 0}\left(\frac{e^x}{x}-\frac{1}{x}\right)$ produces the indeterminant form $\infty-\infty$, and you can't use the rule while it's in this form.

[Hint: Once you find a limit, check it with a graphing calculator, if possible.]

Example find $\lim_{x\to 0}\frac{\sin x}{x}$

1. Check the form by substitution: $\frac{\sin(0)}{0} = \frac{0}{0}$ √ **ok**

2. Apply L'Hôpital's Rule: $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}[\sin x]}{\frac{d}{dx}[x]} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

Example find $\lim_{x\to 0}\frac{e^x - 1}{3x + x^2}$

1. Check the form by substitution: $\frac{e^0 - 1}{3(0)+(0)^2} = \frac{0}{0}$ √ **ok**

2. Apply L'Hôpital's Rule: $\lim_{x\to 0}\frac{e^x - 1}{3x + x^2} = \lim_{x\to 0}\frac{\frac{d}{dx}[e^x - 1]}{\frac{d}{dx}[3x + x^2]} = \lim_{x\to 0}\frac{e^x}{3 + 2x} = \frac{1}{3}$

Example find $\lim_{x\to\infty}\frac{e^x}{x}$

1. Check the form by substitution: $\frac{e^\infty}{\infty} = \frac{\infty}{\infty}$ √ **ok**

2. Apply L'Hôpital's Rule: $\lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}[e^x]}{\frac{d}{dx}[x]} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$

Example find $\lim_{x\to\infty}\frac{\ln x}{x^2}$

1. Check the form by substitution: $\frac{\ln(\infty)}{\infty^2} = \frac{\infty}{\infty}$ √ **ok**

2. Apply L'Hôpital's Rule: $\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x^2]} = \lim_{x\to\infty}\frac{(1/x)}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0$

Example find $\lim\limits_{x\to 0^+} x^2 \cot x$

1. Check the form by substitution: $(0)^2 \bullet \cot(0) = 0 \bullet \infty$ **2nd group form**

2. Rewrite and check: $x^2 \cot x = \dfrac{x^2}{\tan x} \rightarrow \dfrac{(0)^2}{\tan(0)} = \dfrac{0}{0}$ **√ ok**

3. Apply L'Hôpital's Rule: $\lim\limits_{x\to 0^+} \dfrac{x^2}{\tan x} = \lim\limits_{x\to 0^+} \dfrac{\frac{d}{dx}\left[x^2\right]}{\frac{d}{dx}\left[\tan x\right]} = \lim\limits_{x\to 0^+} \dfrac{2x}{\sec^2 x} = \dfrac{2 \bullet 0}{\sec^2(0)} = 0$

Example find $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

1. Check the form by substitution: $\dfrac{1}{0} - \dfrac{1}{(0)^2} = \infty - \infty$ **2nd group form**
2. Rewrite and check: $\dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x-1}{x^2} \rightarrow \dfrac{0-1}{0^2} = \dfrac{-1}{0} = -\infty$

3. L'Hôpital's Rule is not needed here: $\lim\limits_{x\to 0^+} \left(\dfrac{x-1}{x^2}\right) = -\infty$

Example find $\lim\limits_{x\to 0} \left(\csc x - \dfrac{1}{x}\right)$

1. Check the form by substitution: $\csc(0) - \dfrac{1}{0} = \dfrac{1}{\sin(0)} - \dfrac{1}{0} = \infty - \infty$ **2nd group form**

2. Rewrite and check: $\csc x - \dfrac{1}{x} = \dfrac{1}{\sin x} - \dfrac{1}{x} = \dfrac{x - \sin x}{x(\sin x)} \rightarrow \dfrac{0 - \sin(0)}{0(\sin(0))} = \dfrac{0}{0}$ **√ ok**

3. Apply L'Hôpital's Rule: $\lim\limits_{x\to 0} \left(\dfrac{x - \sin x}{x(\sin x)}\right) = \lim\limits_{x\to 0} \left(\dfrac{\frac{d}{dx}\left[x - \sin x\right]}{\frac{d}{dx}\left[x(\sin x)\right]}\right) = \lim\limits_{x\to 0} \dfrac{1 - \cos x}{\sin x + x \cos x}$

4. Check the form by substitution: $\dfrac{1 - \cos(0)}{\sin(0) + (0)\cos(0)} = \dfrac{0}{0}$ **√ ok, but still indeterminant**

5. Apply L'Hôpital's Rule again: $\lim_{x\to 0}\dfrac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0}\dfrac{\frac{d}{dx}[1-\cos x]}{\frac{d}{dx}[\sin x + x\cos x]}$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{\sin(0)}{2\cos(0)-(0)\sin(0)} = \frac{0}{2} = 0$$

Example find $\lim_{x\to 0}\left(e^x + x\right)^{1/x}$

1. Check the form by substitution: $\left(e^0+0\right)^{1/0} = 1^\infty$ **2^{nd} group form**

2. Rewrite using the natural logarithm: $y = \left(e^x + x\right)^{1/x} \rightarrow \ln y = \dfrac{\ln(e^x + x)}{x}$

3. Check the new form by substitution: $\dfrac{\ln(e^0+0)}{0} = \dfrac{\ln 1}{0} = \dfrac{0}{0}$ **√ ok**

Now we'll find the limit of ln y.

4. Apply L'Hôpital's Rule: $\lim_{x\to 0}\dfrac{\ln(e^x+x)}{x} = \lim_{x\to 0}\dfrac{\frac{d}{dx}\left[\ln(e^x+x)\right]}{\frac{d}{dx}[x]}$

$$= \lim_{x\to 0}\frac{\left[\frac{e^x+1}{e^x+x}\right]}{1} = \frac{e^0+1}{e^0+0} = \frac{2}{1} = 2 = \text{limit of } \ln y$$

5. Since the limit of ln y = 2, the limit of $y = e^2$.

Example find $\lim_{x\to 0^+}(2x)^{x/4}$

1. Check the form by substitution: $(2\bullet 0)^{0/4} = 0^0$ **2^{nd} group form**

2. Since x is in the exponent, we'll use logarithms again: $y = (2x)^{x/4} \rightarrow \ln y = \dfrac{x}{4}\bullet\ln(2x)$

3. Check the new form by substitution: $\dfrac{0}{4}\bullet\ln(2\bullet 0) = 0\bullet -\infty$ **2^{nd} group form**

4. Rewrite and check the form: $\dfrac{x}{4}\bullet\ln(2x) = \dfrac{\ln(2x)}{\frac{4}{x}} \rightarrow \dfrac{\ln(2\bullet 0)}{\frac{4}{0}} = \dfrac{-\infty}{\infty}$ **√ ok**

Now we'll find the limit of ln y.

5. Apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\ln(2x)}{\frac{4}{x}} = \lim_{x\to 0^+}\frac{\frac{d}{dx}[\ln(2x)]}{\frac{d}{dx}\left[\frac{4}{x}\right]} = \lim_{x\to 0^+}\frac{\left[\frac{1}{x}\right]}{\left[\frac{-4}{x^2}\right]} = \lim_{x\to 0^+}\frac{-x^2}{4x} = \lim_{x\to 0^+}\frac{-x}{4} = 0 = \text{limit of } \ln y$$

6. Since the limit of ln y = 0, the limit of $y = e^0 = 1$.

<u>Example</u> find $\lim_{x\to 1^+}(x-1)^{\ln x}$

1. Check the form by substitution: $(1-1)^{\ln(1)} = 0^0$ **2^{nd} group form**
2. We'll use logarithms again: $y = (x-1)^{\ln x} \rightarrow \ln y = (\ln x)[\ln(x-1)]$

3. Check the new form by substitution: $(\ln 1)[\ln(1-1)] = (0)[\ln(0)] = 0\bullet-\infty$

2^{nd} group form

4. Rewrite and check the form: $(\ln x)[\ln(x-1)] = \dfrac{\ln(x-1)}{\frac{1}{\ln x}} \rightarrow \dfrac{\ln(1-1)}{\frac{1}{\ln 1}} = \dfrac{\ln 0}{\frac{1}{0}} = \dfrac{-\infty}{\infty}$ **√ ok**

<u>Now we'll find the limit of ln y</u>.

5. Apply L'Hôpital's Rule: $\lim_{x\to 1^+}\dfrac{\ln(x-1)}{(1/\ln x)} = \lim_{x\to 1^+}\dfrac{\frac{d}{dx}[\ln(x-1)]}{\frac{d}{dx}[(1/\ln x)]} = \lim_{x\to 1^+}\dfrac{[1/(x-1)]}{\left[\frac{-1}{(\ln x)^2}\right]\frac{1}{x}}$

$$= \lim_{x\to 1^+}\frac{-x(\ln x)^2}{(x-1)}$$

Let's check again: $\dfrac{-1(\ln 1)^2}{(1-1)} = \dfrac{0}{0}$ **√ ok, but still indeterminant**

6. Apply L'Hôpital's Rule again:

$$\lim_{x\to 1^+}\frac{-x(\ln x)^2}{(x-1)} = \lim_{x\to 1^+}\frac{\frac{d}{dx}\left[-x(\ln x)^2\right]}{\frac{d}{dx}[(x-1)]} = \lim_{x\to 1^+}\frac{-(\ln x)^2 - 2x(\ln x)\left(\frac{1}{x}\right)}{1}$$

$$= \lim_{x\to 1^+}\left[-(\ln x)^2 - 2\ln x\right] = -(\ln 1)^2 - 2\ln(1) = 0 = \text{limit of } \ln y$$

7. Since the limit of ln y = 0, the limit of $y = e^0 = 1$.

You don't always need to use L'Hôpital's Rule – sometimes other techniques are simpler. So occasionally step back and look at it!

<u>Example</u> find $\lim_{x\to\infty}\left[\sqrt{x^2+5x}-x\right]$

1. Check the form by substitution: $\sqrt{\infty^2+5\bullet\infty}-\infty = \infty-\infty$ **2nd group form**

2. To try L'Hôpital's Rule, let's build a quotient of functions by rationalizing the numerator:

$$\left[\sqrt{x^2+5x}-x\right]\frac{\left[\sqrt{x^2+5x}+x\right]}{\left[\sqrt{x^2+5x}+x\right]} = \frac{x^2+5x-x^2}{\sqrt{x^2+5x}+x} = \frac{5x}{\sqrt{x^2+5x}+x}$$

3. Check the new form by substitution: $\frac{5\bullet\infty}{\sqrt{\infty^2+5\bullet\infty}+\infty} = \frac{\infty}{\infty}$ **√ ok**

4. Now it qualifies for L'Hôpital's Rule, but try taking the derivatives of the top and bottom functions and you'll find it just leads to more complications. So let's fall back on the old trick of "dividing out the x":

$$\frac{5x}{\sqrt{x^2+5x}+x} = \frac{5}{\frac{\sqrt{x^2+5x}}{x}+1} = \frac{5}{\frac{\sqrt{x^2+5x}}{\sqrt{x^2}}+1} = \frac{5}{\sqrt{\frac{x^2+5x}{x^2}}+1} = \frac{5}{\sqrt{1+5/x}+1}$$

5. Now take the limit: $\lim_{x\to\infty}\left[\frac{5}{\sqrt{1+5/x}+1}\right] = \frac{5}{\sqrt{1+0}+1} = \frac{5}{2}$

Be creative! Try something, and if it doesn't work, try something else!