Improper Integrals

Improper integrals either have infinity (or negative infinity) as one or both of the limits of integration, or have an integration limit where the function approaches infinity (or negative infinity), or have an internal infinite discontinuity in the interval of integration (a vertical asymptote). Evaluating these integrals requires finding limits as infinity or the discontinuity value is approached.

If *f* is continuous on [a, ∞), then: $$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

If *f* is continuous on (-∞, a], then: $$\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty}\int_b^a f(x)dx$$

If *f* is continuous on (-∞, ∞), then: $$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \quad (c \text{ any real number})$$

If the limit exists, the improper integral **converges**; otherwise it **diverges**. In the third case, both of the integrals on the right have to converge for the integral on the left to converge.

If you're dealing with a rational function that has a vertical asymptote, be careful trying to do integration over an interval containing that asymptote. The end points of the interval may not cause a problem in the integrated function, but will produce a wrong answer if the discontinuity is not accounted for.

If *f* is continuous on [a, b), with an infinite discontinuity at b, then: $$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

If *f* is continuous on (a, b], with an infinite discontinuity at a, then: $$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

If *f* has an infinite discontinuity at *c* on the interval [a, b], then: $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If the limit exists, the improper integral **converges**; otherwise it **diverges**. In the third case, both of the integrals on the right have to converge for the integral on the left to converge.

A special improper integral with an infinite limit is:

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

Evaluate: $\int_0^1 \ln x \, dx$

1. Internal discontinuity? No, but the function approaches -∞ as x approaches 0.

2. $\int_0^1 \ln x \, dx = \lim_{b\to 0^+} \int_b^1 \ln x \, dx = \lim_{b\to 0^+} \left[x \ln x - x\right]_b^1$

$$= \left[(1)\ln(1) - 1\right] - \lim_{b\to 0^+}(b\ln(b) - b) = 0 - 1 - \lim_{b\to 0^+}(b\ln(b)) + 0 = -1 - \lim_{b\to 0^+}(b\ln(b))$$

3. To evaluate $\lim_{b\to 0^+}(b \ln b)$, first check its form by substitution: $(0)(\ln(0)) = 0 \bullet -\infty$

   This is an indeterminate form that requires using L'Hôpital's Rule, after a rewrite.

4. $b \ln(b) = \dfrac{\ln(b)}{(1/b)}$ → check the form: $\dfrac{\ln(0)}{(1/0)} = \dfrac{-\infty}{\infty}$ → qualifies for L'Hôpital's Rule

5. Apply L'Hôpital's Rule: $\lim_{b\to 0^+} \dfrac{\ln(b)}{(1/b)} = \lim_{b\to 0^+} \dfrac{\frac{d}{db}\left[\ln(b)\right]}{\frac{d}{db}\left[(1/b)\right]} = \lim_{b\to 0^+}(-b) = 0$

6. The limit of the integral is now: $-1 - 0 = -1$

Evaluate: $\int_{-1}^{\infty} e^{-x} \, dx$

1. Internal discontinuity? No.

2. $\int_{-1}^{\infty} e^{-x} \, dx = \lim_{b\to\infty} \int_{-1}^{b} e^{-x} \, dx = \lim_{b\to\infty} \left[-e^{-x}\right]_{-1}^{b} = \lim_{b\to\infty}(-e^{-b}) + e^{1} = 0 + e = e$

Evaluate: $\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx$

1. Internal discontinuity? Not in the interval of integration.

2. $\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{\sqrt{x}} \, dx = \lim_{b\to\infty} \left[2\sqrt{x}\right]_1^{b} = \lim_{b\to\infty} \left[2\sqrt{b}\right] - 2\sqrt{1} = \infty$ diverges

Evaluate: $\int_{-1}^{2} \frac{1}{x^3}\,dx$

1. Internal discontinuity? Yes, vertical asymptote at x = 0.

2. Split the integral around the asymptote: $\int_{-1}^{2} \frac{1}{x^3}\,dx = \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^3}\,dx + \lim_{c\to 0^+} \int_{c}^{2} \frac{1}{x^3}\,dx$

$$= \lim_{c\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{c} + \lim_{c\to 0^+}\left[-\frac{1}{2x^2}\right]_{c}^{2} = \left[\lim_{c\to 0^-}\left[-\frac{1}{2c^2}\right] + \frac{1}{2(-1)^2}\right] + \left[-\frac{1}{2(2)^2} - \lim_{c\to 0^+}\left[-\frac{1}{2c^2}\right]\right]$$

$$= \left[-\infty + 1/2\right] + \left[-1/8 + \infty\right] \qquad \text{diverges}$$

Note that the infinities do not cancel! When an integral is split, if either of the right-hand integrals diverge, then the original integral diverges. In this case, both of them diverge.

Evaluate: $\int_{0}^{4} \frac{6}{\sqrt{4-x}}\,dx$

1. Internal discontinuity? No, but the function approaches infinity as x approaches 4.

2. $\int_{0}^{4} \frac{6}{\sqrt{4-x}}\,dx = \lim_{a\to 4^-} \int_{0}^{a} \frac{6}{\sqrt{4-x}}\,dx = \lim_{a\to 4^-}\left[-12\sqrt{4-x}\right]_0^a = -12\sqrt{4-4} + 12\sqrt{4-0} = 24$

[Hint: do the integration using *u* substitution, with u = 4 – x.]

Evaluate: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

1. Internal discontinuity? No, but the integral must be split because of the double infinite limits, so we'll split it around zero for convenience.

2. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to -\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to \infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx = \lim_{a\to -\infty}\left[\arctan x\right]_a^0 + \lim_{b\to\infty}\left[\arctan x\right]_0^b$

$$= 0 - (-\pi/2) + \pi/2 - 0 = \pi$$

Evaluate: $\int_0^1 \frac{1}{x}\,dx$

1. Internal discontinuity? No, but the function approaches infinity as x approaches 0.

2. $\int_0^1 \frac{1}{x}\,dx = \lim_{b\to 0^+} \int_b^1 \frac{1}{x}\,dx = \lim_{b\to 0^+} \left[\ln x\right]_b^1 = \ln(1) - \lim_{b\to 0^+}\left[\ln b\right] = 0 - (-\infty) = \infty$ diverges

Evaluate: $\int_2^\infty x\,e^{-x}\,dx$

1. Internal discontinuity? No.

2. $\int_2^\infty x\,e^{-x}\,dx = \lim_{b\to\infty}\int_2^b x\,e^{-x}\,dx = \lim_{b\to\infty}\left[-x\,e^{-x} - e^{-x}\right]_2^b$

$$= \lim_{b\to\infty}\left[\frac{-b}{e^b} - \frac{1}{e^b}\right] + \frac{2}{e^2} + \frac{1}{e^2} = \lim_{b\to\infty}\left[\frac{-b}{e^b}\right] + \frac{3}{e^2}$$

3. To evaluate $\lim_{b\to\infty}\left[\frac{-b}{e^b}\right]$, first check its form by substitution: $\frac{-\infty}{e^\infty} = \frac{-\infty}{\infty}$

4. Apply L'Hôpital's Rule: $\lim_{b\to\infty}\left[\frac{-b}{e^b}\right] = \lim_{b\to\infty}\left[\frac{\frac{d}{db}(-b)}{\frac{d}{db}(e^b)}\right] = \lim_{b\to\infty}\left[\frac{-1}{e^b}\right] = \frac{-1}{e^\infty} = 0$

5. The limit of the integral is now: $0 + \frac{3}{e^2} = \frac{3}{e^2}$

Evaluate: $\int_0^{\pi/2} \sec^2\theta\,d\theta$

1. Internal discontinuity? No, but the function approaches infinity as θ approaches π/2.

2. $\int_0^{\pi/2} \sec^2\theta\,d\theta = \lim_{a\to\frac{\pi}{2}^-}\int_0^a \sec^2\theta\,d\theta = \lim_{a\to\frac{\pi}{2}^-}\left[\tan\theta\right]_0^a = \lim_{a\to\frac{\pi}{2}^-}\left[\tan(a)\right] - \tan(0)$

$= \infty - 0 = \infty$ diverges

Evaluate: $\displaystyle\int_0^{\infty} \frac{6}{\sqrt{x}(x+4)}\,dx$

1. Internal discontinuity? No, but the integral is doubly improper, so we need to split it.

2. Split the integral: $\displaystyle\int_0^{\infty} \frac{6}{\sqrt{x}(x+4)}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{6}{\sqrt{x}(x+4)}\,dx + \lim_{b\to\infty}\int_1^b \frac{6}{\sqrt{x}(x+4)}\,dx$

$$= \lim_{a\to 0^+}\left[6\arctan\frac{\sqrt{x}}{2}\right]_a^1 + \lim_{b\to\infty}\left[6\arctan\frac{\sqrt{x}}{2}\right]_1^b$$

[Hint: use $u = \sqrt{x}$ to do the integration]

$$= \cancel{6\arctan}\left(\frac{1}{2}\right) - 6\arctan(0) + \lim_{b\to\infty}\left[6\arctan\frac{\sqrt{b}}{2}\right] - \cancel{6\arctan}\left(\frac{1}{2}\right)$$

$$= 0 + 6\left(\frac{\pi}{2}\right) = 3\pi$$

Evaluate: $\displaystyle\int_0^{\infty} \sin\theta\,d\theta$

1. Internal discontinuity? No

2. $\displaystyle\int_0^{\infty} \sin\theta\,d\theta = \lim_{a\to\infty}\int_0^a \sin\theta\,d\theta = \lim_{a\to\infty}\left[-\cos\theta\right]_0^a = \lim_{a\to\infty}\left[-\cos(a)\right] - (-\cos(0))$

$\displaystyle\lim_{a\to\infty}\left[-\cos(a)\right] + 1$ diverges, because –cos(a) never reaches a limit

Find the volume of the solid made by rotating the curve $1/x^2$ around the x-axis, with limits from 1 to infinity.

The formula for volume is: $\displaystyle V = \pi\int_1^{\infty}\left(\frac{1}{x^2}\right)^2 dx = \pi\int_1^{\infty}\frac{1}{x^4}\,dx$

Using the special improper integral $\displaystyle\int_1^{\infty}\frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p>1 \\ \text{diverges}, & \text{if } p\le 1 \end{cases}$, $\displaystyle V = \pi\left(\frac{1}{4-1}\right) = \frac{\pi}{3}$