Implicit Differentiation and Related Rates
(MAT 133, MAT 201, MAT 218)

You have already done many problems like this one:

Find $\frac{dy}{dx}$ if $y = 3x^2 + (x^3 - 7x)^5$

Let's look at the answer to this problem and make note of several important things.

$$y = 3x^2 + (x^3 - 7x)^5$$

$$\frac{dy}{dx} = 6x + 5(x^3 - 7x)^4 (3x^2 - 7)$$

↓ the derivative of **y** with respect to **x** is $\frac{dy}{dx}$.

↓ Since we are differentiating with respect to **x**, the derivative of this term can be found by simply using the power rule.

↓ Differentiating this term requires not only the power rule, but also the chain rule. Why? Because the base was **NOT X**. We had to multiply by **the derivative of** the base.

Up to this point, all of the functions you have worked with have been conveniently solved for y in terms of x. (These are called **explicitly** defined functions.) An **implicitly** defined function (or relation) is not solved for y. An example would be:

$$y^3 + y - 2x^2y^5 = 8 + 3x^4$$

Solving an equation like this for y can be difficult (if not impossible). Can we still find $\frac{dy}{dx}$?

Yes, if we remember that:

1) the derivative of y is $\frac{dy}{dx}$.
2) using a differentiation rule on something that is "not x" requires the chain rule.
3) y is "not x"!

Let's find $\frac{dy}{dx}$. $y^3 + y - 2x^2y^5 = 8 + 3x^4$

Differentiate each term with respect to x.

Uh-oh! Need Product Rule!

derivative of y^3 + derivative of y – [$2x^2$ • derivative of y^5 + y^5 • derivative of $2x^2$] = derivative of 8 + derivative of $3x^4$

$$3y^2 \cdot \frac{dy}{dx} + 1 \cdot \frac{dy}{dx} - [2x^2 \cdot 5y^4 \cdot \frac{dy}{dx} + y^5 \cdot 4x] = 0 + 12x^3$$

$3y^2$	$\frac{dy}{dx}$	1	$\frac{dy}{dx}$	$5y^4$	$\frac{dy}{dx}$	$4x$	0	$12x^3$
↑ power rule	↑ chain rule (y is "not x"!)	↑ power rule	↑ chain rule (y is "not x"!)	↑ power rule	↑ chain rule	↑ no chain rule necessary! (no y involved)	↑ derivative of a constant	↑ no chain rule necessary

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} - 10x^2y^4\frac{dy}{dx} - 4xy^5 = 12x^3$$

Now we can "solve" this equation for the "unknown" $\frac{dy}{dx}$. Keep all terms with $\frac{dy}{dx}$ in them on one side. Move other terms to the opposite side.

$$3y^2 \frac{dy}{dx} + \frac{dy}{dx} - 10x^2y^4\frac{dy}{dx} = 12x^3 + 4xy^5$$

Factor out $\frac{dy}{dx}$ and solve.

$$\frac{dy}{dx}(3y^2 + 1 - 10x^2y^4) = 12x^3 + 4xy^5$$

$$\frac{dy}{dx} = \frac{12x^3 + 4xy^5}{3y^2 + 1 - 10x^2y^4}$$

Unfortunately, our expression for $\frac{dy}{dx}$ does still contain the variable y. We'll just have to live with that. Theoretically, we could solve the original equation for y in terms of x and then replace the y's with expressions involving only x. However, if the original could have been easily solved for y, we wouldn't have done the problem with implicit differentiation in the first place! Fortunately, if we are interested in the derivative (or slope of the tangent line) at a particular point with known coordinates, there is no problem. Just plug the x and y coordinates into your expression for $\frac{dy}{dx}$.

Can we find the second derivative (or third or fourth…) using implicit differentiation? Yes. The above example will be too complicated so we will start with a new example.

$$x^2 + y^2 = y$$

$$2x + 2y \bullet \frac{dy}{dx} = 1 \bullet \frac{dy}{dx}$$

$$2x = \frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$2x = \frac{dy}{dx}(1-2y)$$

$$\frac{dy}{dx} = \frac{2x}{1-2y}$$

$$\frac{d^2y}{dx^2} = \overbrace{\frac{(1-2y)(2)-(2x)\left(-2\frac{dy}{dx}\right)}{(1-2y)^2}}^{\text{quotient rule}}$$

↑
the second derivative
is the derivative of
the first derivative

Replace $\frac{dy}{dx}$ with $\frac{2x}{1-2y}$ (from above).

$$\frac{d^2y}{dx^2} = \frac{2(1-2y)+4x\left(\frac{2x}{1-2y}\right)}{(1-2y)^2}$$

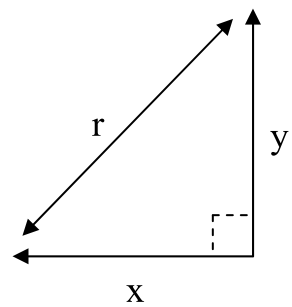
NOTE: These expressions can be difficult to simplify algebraically. If you are only interested in the value of a derivative at a known point, don't waste your time trying to simplify. Plug in your x and y coordinates. Arithmetic is easier than algebra!

Related Rate Problems

The ability to differentiate implicitly is a necessary skill for working with a whole class of problems know as **RELATED RATE PROBLEMS.** Here is an example of a classic related rate problem:

> Two cars are at the same intersection at 12:00 noon. One car heads north at a rate of 20 miles per hour. The other car heads west at a rate of 15 miles per hour. How fast is the distance between the cars changing at 2:00 pm?

We can visualize this situation as an "expanding triangle". Since all sides of the triangle are changing, we will label them with arbitrary variables. We can see from the diagram that there is a mathematical relationship between the 3 distances x, y, and r. They can be related by the Pythagorean Theorem: $x^2 + y^2 = r^2$. But there is a fourth "invisible" variable in this problem – time. The variables x, y, and r are also related to this fourth variable, time.



Let's take a break from this problem for a minute and re-visit the concept of implicit differentiation.

When differentiating with respect to x, we need to use the chain rule when differentiating something that is not x. What if we were differentiating something with respect to t (time), or any other variable for that matter? We would need the chain rule for anything that was not t.

The derivative of a term like $4y^3$ would be $12y^2$ • (the derivative of y with respect to t) = $12y^2 \bullet \frac{dy}{dt}$.

The derivative of a term like 4x would by 4 • (the derivative of x with respect to t) = $4 \bullet \frac{dx}{dt}$.

So, we could differentiate an equation implicitly with respect to t instead of x. Differentiating any term with variables that are not t would require the chain rule.

Let's go back to our related rate problem.

We know that $x^2 + y^2 = r^2$. If we differentiate this equation implicitly with respect to t (time) we would get:

$$2x \bullet \frac{dx}{dt} + 2y \bullet \frac{dy}{dt} = 2r \bullet \frac{dr}{dt}$$

Remember that derivatives are rates of change:

$\frac{dy}{dt}$ = the rate of change in y with respect to time
= the rate of change in the northbound car's distance with respect to time
= its speed!

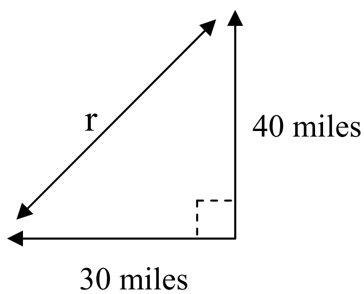
$\frac{dx}{dt}$ = the rate of change in x with respect to time
= the rate of change in the westbound car's distance with respect to time
= its speed!

$$\frac{dr}{dt} = \text{the rate of change in the distance between the 2 cars with respect to time}$$
$$= \text{the speed with which they are moving apart!}$$

So we now have an equation that "relates rates": $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

We know the speeds of the 2 cars ($\frac{dx}{dt}$ and $\frac{dy}{dt}$), so $2x(15) + 2y(20) = 2r\frac{dr}{dt}$

If we could find out what x, y, and r are (at 2:00 pm) we can solve this equation for $\frac{dr}{dt}$ (how fast the distance is changing between the 2 cars at 2:00 pm). At 2:00 pm, each car has traveled for 2 hours, one at 15 mph and the other at 20 mph, so the diagram looks like this:



We can find the distance between the cars (r) by the Pythagorean Theorem: $30^2 + 40^2 = r^2 \rightarrow r = 50$.

Back to our "related rates" equation:

$$2x(15) + 2y(20) = 2r\frac{dr}{dt}$$

$$2(30)(15) + 2(40)(20) = 2(50)\frac{dr}{dt}$$

Solve for $\frac{dr}{dt}$: $\frac{dr}{dt} = 25$

At 2:00 pm, the distance between the 2 cars is increasing at the rate of 25 mph.

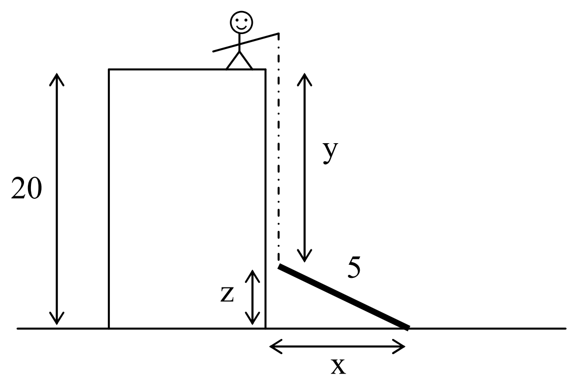
Before we proceed to another example, here are a few important notes:

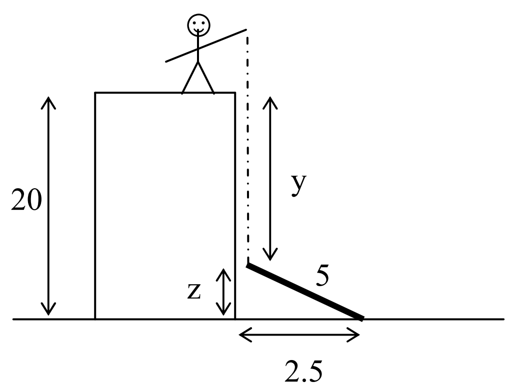
1) If you are not given an equation that relates the variables, you will have to write your own. Use whatever applies to the situation (Pythagorean Theorem, geometry formulas, trigonometric identities, similar triangles, etc.).

2) DO NOT plug in numbers to replace variables until AFTER you have done your differentiation and have a related rate equation.

3) The equation you need to differentiate may have "extra" variables in it. Figure out some way to replace those BEFORE you differentiate (or you will end up with too many variables and rates in your related rate equation and you will not be able to solve for the rate you are looking for).

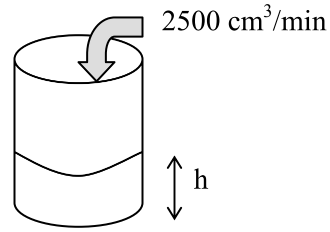
4) An increasing variable will have a positive rate of change. A decreasing variable will have a negative rate of change.

STEPS TO FOLLOW TO SOLVE A RELATED RATE PROBLEM	ONE MORE EXAMPLE
1. Read the problem and concentrate on understanding the situation. Draw a diagram (if relevant). Label things that change with variables.	A construction worker pulls a 5 meter plank that is lying on the ground up the side of a 20 meter building by means of a rope tied to one end of the plank. The worker pulls the rope straight up at a rate of 0.15 meters per second. How fast is the other end of the plank sliding along the ground when it is 2.5 meters from the wall of the building? 
2. Using the variables from your diagram (or equation, if the equation that relates the variables is given to you), read the problem again carefully and list the "rates" that are given in the problem (including the "rate" that you are asked to find).	$\frac{dy}{dt} = -0.15$ "…pulls the rope..at a rate…" **Note**: y is <u>decreasing</u> as the worker pulls. The rate is <u>negative</u>. $\frac{dx}{dt}$ = unknown "…how fast..other end..sliding…?"
3. Look at the "top" variables on your rate list. You need an equation that relates those variables. Replace any "extra" variables by whatever means possible.	We need an equation that relates y and x. From the diagram, we see 2 relationships: $z^2 + x^2 = 5^2$ and $y + z = 20$ z is an "extra" variable. Use the first equation and replace z with (20 – y) from the second equation: $(20 - y)^2 + x^2 = 5^2$

4. Differentiate your equation with respect to time (t). (NOTE: not <u>all</u> related rate problems deal with time. You may have to differentiate with respect to some other variable.)	This equation will be easier to differentiate if we simplify it first: $400 - 40y + y^2 + x^2 = 5^2$ Differentiate with respect to t. $0 - 40 \bullet \frac{dy}{dt} + 2y \bullet \frac{dy}{dt} + 2x \bullet \frac{dx}{dt} = 0$
5. Draw another diagram (if necessary) that shows the situation for the specific moment asked about in the original problem and calculate as many variable values as possible.	"…when it is 2.5 meters from the wall…"  From $z^2 + x^2 = 5^2$: $z^2 + (2.5)^2 = 25$ $z^2 = 18.75$ $z = 4.33$ From $y + z = 20$: $y + 4.33 = 20$ $y = 15.67$
6. Plug all known values and rates in to your related rate equation and solve for the unknown rate.	$-40\frac{dy}{dt} + 2y\frac{dy}{dt} + 2x\frac{dx}{dt} = 0$ $-40\,(-0.15) + 2(15.67)(-0.15) + 2(2.5)\frac{dx}{dt} = 0$ $6 \quad - \quad 4.701 \quad + 5\frac{dx}{dt} = 0$ $5\frac{dx}{dt} = -1.299$ $\frac{dx}{dt} = -0.26$
7. Answer the question.	The other end of the plank is sliding toward the wall at a rate of 0.26 meters per second. (x is <u>decreasing</u> so the rate came out <u>negative</u>.)

Some Solved Problems

1. A pipe is filling a cylindrical tank at the rate of 2500 cm^3 per minute. If the radius of the tank is 25 cm, how fast is the height of the water changing?



Given: $V = \pi r^2 h$

1) What's changing? Volume and height.

So our rates are: $\frac{dV}{dt} = 2500\text{ cm}^3/\text{min}$

and $\frac{dh}{dt}$ = unknown. r is constant at 25 cm. $V = \pi(25)^2 h$

2) Replace r and differentiate with respect to time. $\frac{dV}{dt} = 625\pi\frac{dh}{dt}$

3) Plug in known rates and solve. $2500 = 625\pi\frac{dh}{dt}$

4) Answer: The height is increasing by 1.27 cm/min. $\frac{dh}{dt} = 1.27$

2. The price of beans *p* (in dollars per bag) is related to the weekly supply *x* (in thousands of bags) by the equation $500p^2 - x^2 = 100$. If 20,000 bags of beans are available for a certain week, and the price is falling at the rate of 2 cents per bag per week, at what rate is the supply falling?

1) What's changing? Price and supply. The rates are: $\frac{dp}{dt} = -\$0.02$ and $\frac{dx}{dt}$ = unknown.

2) We have the equation that relates p and x, $500p^2 - x^2 = 100$

so differentiate with respect to time. $500(2p)\frac{dp}{dt} - (2x)\frac{dx}{dt} = 0$

3) When x = 20, what is p? Use the original equation. $500p^2 - 20^2 = 100$

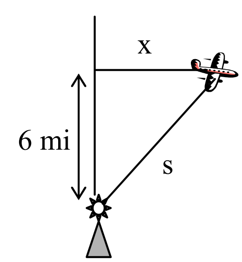
$p = 1$

4) Plug in known quantities and rates into $500(2 \cdot 1)(-0.02) - (2 \cdot 20)\frac{dx}{dt} = 0$

the differentiated equation and solve for $\frac{dx}{dt}$. $\frac{dx}{dt} = -0.5$

5) Answer: The supply is falling at a rate of 500 bags/week

3. An airplane flying at an altitude of 6 miles passes directly over a radar antenna. When the plane is 12 miles away (slant distance: s = 12), the radar detects that the distance s is changing at 220 miles per hour. What is the speed of the plane?



1) What's changing? x and s. The rates are: $\frac{dx}{dt}$ = unknown and $\frac{ds}{dt} = 220$ mph.

2) How are x and s related? By the Pythagorean Theorem: $x^2 + 6^2 = s^2$

3) Differentiate with respect to time: $2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

4) When s = 12, what is x? $x^2 + 6^2 = 12^2 \rightarrow x = 10.39$

5) Plug in known quantities and rates and solve $2(10.39)\frac{dx}{dt} = 2(12)(220)$

for $\frac{dx}{dt}$ (the speed of the plane). $\frac{dx}{dt} = 254.1$ mph