EXTREMA OF FUNCTIONS OF TWO VARIABLES

When asked to find the extrema of a function f(x,y) we are really being asked to find the maximum and/or minimum values of f(x,y), if they exist. Once we find these values we have to determine whether the values are relative or absolute. To do this we need to know the difference between a relative maximum (minimum) and an absolute maximum (minimum). The distinction between relative and absolute boils down to the distinction between an open region, disc, or ball and a domain. Let's say that at the point (x_0, y_0), the function $f(x_0, y_0)$ is greater than or equal to f(x,y) for all other values of (x,y) in an open region, disc, or ball containing (x_0, y_0). We say f(x,y) has a relative maximum at (x_0, y_0). Suppose that $f(x_0, y_0)$ is greater than or equal to f(x,y) for all other values of (x,y) in the domain of f. Then we say f(x,y) has an absolute maximum at (x_0, y_0). Note that a point that satisfies both of these conditions is both a relative maximum and an absolute maximum at the same time. Sometimes a relative maximum is called a local maximum.

To find the extrema of a function of two variables, f(x,y), we first find a critical point (x_0, y_0). The point (x_0, y_0) is a critical point if $f_x(x_0, y_0) = 0$ *and* $f_y(x_0, y_0) = 0$. If $f_x(x_0, y_0)$ *or* $f_y(x_0, y_0)$ does not exist , then (x_0, y_0) is a critical point. Relative extrema occur only at critical points.

<u>Example</u>: Find any relative extrema of $f(x, y) = x^2 + 5y^2 - 4x + 10$.

We first take partial derivatives.

$$f_x = 2x - 4, \quad f_y = 10y$$
$$f_x = 0 \Rightarrow x = 2, \quad f_y = 0 \Rightarrow y = 0$$

(2,0) is a critical point, and f(2,0)=6. We need to check to see if (2,0) gives an extrema. Completing the square in f(x,y) gives

$$x^2 - 4x + 4 + 5y^2 + 10 - 4 = (x-2)^2 + 5y^2 + 6$$

No matter what values we select for x and y, $(x-2)^2$ *and* y^2 will always be positive. So,

$$(x-2)^2 + 5y^2 + 6 \geq 6 \Rightarrow f(x, y) \geq f(2,0) \text{ for all } (x, y).$$

(2,0) is a relative minimum.

<u>Example</u>: Find the extrema of $f(x, y) = yx^2 - x + 3$.

Taking partial derivatives, we have $f_x = 2xy - 1,\ f_y = x^2$

Setting these equal to zero, no value of x or y will satisfy $x^2 = 0$ *and* $2xy = 1$. There are no critical points.

In the above examples we used algebraic means to determine whether critical points gave maximum or minimum values of the function. With more complicated functions we rely on the second partials test:

1. For the function $f(x,y)$, find (x_0, y_0) so that $f_x(x_0,y_0)=0, f_y(x_0,y_0)=0.$

2. Compute $f_{xx}(x_0,y_0),\ f_{yy}(x_0,y_0),\ and\ f_{xy}(x_0,y_0).$

3. Set $D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2$

Then, if

$D \succ 0\ \ and\ \ f_{xx}(x_0,y_0) \succ 0,\ \ f(x,y)$ has a relative minimum at (x_0,y_0)
$D \succ 0\ and\ \ f_{xx}(x_0,y_0) \prec 0,\ \ f(x,y)$ has a relative maximum at (x_0,y_0)
$D \prec 0\ \ then\ \ (x_0, y_0, f(x_0,y_0))$ is a saddle point
$D = 0$ test is inconclusive

Mnemonic to remember is the determinant:

$$\begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix}$$

<u>Example</u>: Examine $f(x,y) = 3x^2 + 3xy + y^2 + 3x - 4$ for relative extrema and saddle points. We have $f_x = 6x + 3y + 3,\ and\ \ f_y = 3x + 2y.$ Setting both of these equal to zero and solving gives x=-2, y=3. Then (-2,3) is a critical point. We then have $f_{xx} = 6,\ f_{yy} = 2,\ and\ \ f_{xy} = 3.$

We now compute $D = f_{xx}(-2,3) f_{yy}(-2,3) - [f_{xy}(-2,3)]^2 = 12 - 3^2 = 3 \succ 0$

Since both D and f_{xx} are both greater than zero, (-2,3) gives a relative minimum. f(-2,3)=5.

Example: Examine $f(x, y) = 3x^2 - 3y^2 - 4x + 2y - 8$ for relative extrema and saddle points. We have $f_x = 6x - 4,\ and\ \ f_y = -6y + 2.$ Setting $f_x = f_y = 0$

And solving gives x=2/3, y=1/3. Then (2/3,1/3) is a critical point.

$$f_{xx} = 6,\ f_{yy} = -6,\ f_{xy} = 0$$
$$D = f_{xx}(2/3,1/3) f_{yy}(2/3,1/3) - (0)^2 = -36$$

Since D is less than zero, (2/3,1/3,-9) is a saddle point.

Problems: Examine the following functions for extrema and saddle points.

1. $f(x, y) = 4x^2 + y^2$
2. $f(x, y) = 3xy - x^3 - y^3$
3. $f(x, y) = e^{-x^2 - y^2}$
4. $f(x, y) = x^2 + 2y$
5. $f(x, y) = \sin x + \sin y$

Answers:

1. Critical point (0,0), absolute minimum.
2. Critical point (1,1), relative maximum
3. Critical point (0,0), absolute maximum
4. None
5. Critical point $(\pi/2, \pi/2)$, relative maximum