DIRECTIONAL DERIVATIVES AND GRADIENTS

Suppose you are located on a surface, f(x,y,z) at the point (a,b,c), and below in the xy-plane is a unit vector u with its tail at the origin pointing in some direction. If we erect a vertical plane through the point (a,b,c) which is parallel to the unit vector u, the plane will intersect the surface in a curve.

The slope of the tangent to this curve at the point (a,b,c) in the vertical plane is the directional derivative. It tells you how steep the surface is at (a,b,c) in the same direction as the unit vector u is pointing. There are two ways in which to compute this derivative.

The first method is based on the angle that the unit vector in the xy-plane makes with the positive x-axis. In this case the directional derivative of f(x,y,z) at the point (a,b,c) in the same direction as the unit vector is:

$$D_u f(x,y,z) = f_x(a,b,c)\text{Cos}\theta + f_y(a,b,c)\text{Sin}\theta$$

where θ is the angle that the unit vector makes with the positive x-axis

Example:

Find the directional derivative of $\mathrm{f}(\mathrm{x,y}) = x^2 - y^2$ at the point $(4,2)$ in the direction of a unit vector that makes an angle of $\frac{\pi}{3}$ with the positive x - axis.

$$f(x,y) = x^2 - y^2,\ f_x(x,y) = 2x,\ f_y(x,y) = -2y$$

$$f_x(4,2) = 8,\ f_y(4,2) = -4,\ \cos\frac{\pi}{3} = \frac{1}{2},\ \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$D_u f(x,y) = f_x(4,2)\cos\frac{\pi}{3} + f_y(4,2)\sin\frac{\pi}{3}$$

$$= (8)\left(\frac{1}{2}\right) + (-4)\left(\frac{\sqrt{3}}{2}\right) = 4 - 2\sqrt{3} \cong 0.54$$

Note that the directional derivative is a scalar!

Example: Find the directional derivative of the surface

$f(x,y) = x^3 + 2y^2 - 4$ at the point (2,2) in the same direction as the vector $\mathrm{v} = 3\mathrm{i} + 4\mathrm{j}$.

$f_x(x,y) = 3x^2,\ f_y(x,y) = 4y,\ f_x(2,2) = 12;\ f_y(2,2) = 8$

A unit vector u in the same direction as $\mathrm{v} = 3\mathrm{i} + 4\mathrm{i}$ is $\mathrm{u} = \frac{3}{5}i + \frac{4}{5}j$, so $\cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5}$

$$D_u f(x,y) = (12)\left(\frac{3}{5}\right) + (8)\left(\frac{4}{5}\right) = \frac{68}{5} = 13.6$$

The second method requires two new concepts: the **gradient** of a function, and a **vector differential operator.** The Gradient of f is a vector function, written as ∇f, where ∇ is read as “del”. If f is a function of x,y, and z then the gradient of f is $\nabla f = \langle f_x, f_y, f_z \rangle$

Example: $f\left(x,y,z\right)=x^2+y^2+2zy,$ what is ∇f?

$$f_x=2x,\ f_y=2y+2z,\ f_z=2y$$
$$\nabla f\left(x,y,z\right)=\langle 2x,2y+2z,2y\rangle$$
What is ∇f at $\left(3,1,2\right)$?
$$\nabla f\left(3,1,2\right)=\langle\left(2\right)\left(3\right),\left(2\right)\left(1\right)+\left(2\right)\left(2\right),\left(2\right)\left(1\right)\rangle=\langle 6,6,6\rangle$$
$$or\ \ 6i+6j+6k$$

The operator

∇ can be written $\nabla=\langle\frac{\delta}{\delta \mathrm{x}},\frac{\delta}{\delta y},\frac{\delta}{\delta z}\rangle$

Then operating on f with ∇ would be $\nabla \mathrm{f}=\langle\frac{\delta \mathrm{f}}{\delta \mathrm{x}},\frac{\delta f}{\delta y},\frac{\delta f}{\delta z}\rangle=\langle f_x,f_y,f_z\rangle$

We can now find the directional derivative of f in the direction of a unit vector u by taking the dot product of the gradient of f(x,y) and the unit vector:

$$D_uf\left(x,y\right)=\nabla f\left(x,y\right)\cdot u$$

Example: Find the directional derivative of the surface $f\left(x,y\right)=x^3+2y^2-4$ at the point (2,2) in the same direction as the vector v=3i+4j. A unit vector u for the vector v is u=$\frac{3}{5}i+\frac{4}{5}j$.

$$\nabla f=\langle f_x,f_y\rangle=\langle 3x^2,4y\rangle\quad \nabla f\left(2,2\right)=\langle 12,8\rangle$$
$$D_uf\left(x,y\right)=\nabla f\cdot u=\langle 12,8\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle=\frac{36}{5}+\frac{32}{5}=13.6$$

Example: $f(x,y,z) = x^2 + y^2 + 2yz$ What is the directional derivative at the point (3,1,2) in the direction of the vector v=5i-12j+k ?

A unit vector u for the vector v is given by u $= \dfrac{\langle \mathrm{v} \rangle}{\|\mathrm{v}\|} = \dfrac{\langle 5,-12,1 \rangle}{\sqrt{25+144+1}} = \dfrac{\langle 5,-12,1 \rangle}{\sqrt{170}}$

$\nabla f = \langle 2x, 2y+2z, 2y \rangle$, and at (3,1,2) $\nabla \mathrm{f} = \langle 6,6,2 \rangle$, so we have

$$\mathrm{D_u} f(x,y,z) = \langle 6,6,2 \rangle \cdot \langle \frac{5}{\sqrt{170}}, \frac{-12}{\sqrt{170}}, \frac{1}{\sqrt{170}} \rangle = \frac{30-72+2}{\sqrt{170}} \cong -3.1$$

The directional derivative is a dot product $D_u f = \nabla f \cdot u$, so we can write it as

$D_u f = \|u\| \|\nabla f\| Cos\theta$ where θ is the angle between the gradient and the unit vector u. Since u is a unit vector $\|u\| = 1$ and we have

$$D_u = \|\nabla f\| Cos\theta$$

Now the value of $Cos\theta$ will always be between -1 and +1. When $Cos\theta = 1,\ \theta = 0.$ So the gradient will be in the same direction as the unit vector, and the directional derivative will be at its maximum value.

If $Cos\theta = -1, \theta = \pi,$ and the gradient will be in the opposite diection to the unit vector, and the directional derivative will be at its minimum. In other words, the magnitude of the gradient will equal the maximum value of the directional derivative.

Example: $f(x,y) = x^2 + 2xy + 5$ What is the maximum and minimum directional derivative at the point (1,2)?

$\nabla f = \langle 2x+2y, 2x \rangle$, at the point $(1,2)$ $\nabla f = \langle 5,2 \rangle$

$\|\nabla f\| = \sqrt{25+4} = \sqrt{29}$

Maximum directional derivative = $\sqrt{29}$

Minimum directional derivative = $-\sqrt{29}$

Heat-Seeking Path (Particle or Bug):

The key to solving Heat-Seeking Path problems is to realize that at any point on the path travelled by the particle or bug, the tangent to the curve of the path always points in the same direction as the gradient

Suppose we have a heat-seeking particle which is located at a point (3,3) on a metal plate whose temperature field is given by $T(x,y) = 200 - x^2 - 2y^2$.

We are asked to find the path travelled by the particle.

First, we find the gradient of T(x,y). $\nabla T(x,y) = T_x(x,y)i + T_y(x,y)j$

Next, the position function for the path in vector form is r(t)=x(t)i+y(t)j. The equation of the tangent to the curve f(x,y) in vector form is given by

$$r'(t) = \frac{dx}{dt}i + \frac{dy}{dt}j.$$

Setting this tangent equal to the gradient gives: $-2xi - 4yj = \frac{dx}{dt}i + \frac{dy}{dt}j$

From which we get: $\frac{dx}{dt} = -k2x \quad and \quad \frac{dy}{dt} = -k4y.$

Then, $\dfrac{dx}{2x} = -kdt = \dfrac{dy}{4y}$

or $x^2 = cy$ and since the point (3,3) lies on this curve we get $y = \dfrac{x^2}{3}$

Exercises:

1. Find the directional derivative of f(x,y) in the same direction as v at the point indicated:

$a.\ 4x + 3y \qquad v = \langle 1,-1 \rangle \qquad (x,y)$

$b.\ x^2 + y^2 \qquad v = \langle 3,1 \rangle \qquad (1,-1)$

$c.\ x^2 + y^2 - z \qquad v = \langle 1,1,1 \rangle \qquad \langle -1,1,-1 \rangle$

$d.\ 3\varepsilon^{xyz} \qquad v = \langle 1,1,1 \rangle \qquad (2,1,-1)$

2. Find the gradient of f at the given point.

$a.\ 3x^2 + y^2 \qquad (2,0)$

$b.\ 4x^2 - 5y^2 \qquad (1,1)$

$c.\ Sin\left(\dfrac{x}{y}\right) \qquad (\pi,4)$

$d.\ x^2 + 2yz - z^2 \qquad (0,1)$

3. The temperature at a point (x,y) on a griddle is given by $f(x,y) = 10 - 2x^2 - y^2$. Find the maximum rate of change of temperature at (2,3) in the direction of $\langle 1,1 \rangle$.

4. Find the path of an ant which is dropped on a hot griddle sat the point (2,3) if the temperature at a point (x,y) is given by $15-2x^2+y^2$ if the ant moves in the coolest direction.

Answers:

1a. $\frac{\sqrt{2}}{2}$

b. $2\sqrt{10}$

c. $-\frac{\sqrt{3}}{3}$

d. $-\frac{\sqrt{3}}{\varepsilon^2}$

2.

$a.\ \langle 12,0\rangle$

$b.\ \langle 8,-10\rangle$

$c.\ \frac{\sqrt{2}}{8}\left\langle 1,-\frac{\pi}{4}\right\rangle$

$d.\ \langle 0,2,0\rangle$

3. 10

4. $x=\frac{18}{y^2}$