DIFFERENTIALS AND INCREMENTS

MAT 201 - Differentials

Tangent line (linear) approximation of a graph

The tangent line at a point on a graph will match the graph to an accuracy of several decimal places over a certain (usually short) distance on the graph. The equation of the tangent line at point (c, f(c)) is given by: **$y = f(c) + f'(c)(x - c)$**. As x approaches c, the limit of y is f(c).

<u>Example</u>: find the equation of the tangent line T for $y = \sqrt{x}$ at $(2, \sqrt{2})$ and approximate the function values near x = 2.

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad T(x) = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-2) = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	**1.9**	**1.99**	**2**	**2.01**	**2.1**
f(x)	1.378405	1.410674	1.414214	1.417745	1.449138
T(x)	1.378858	1.410678	1.414214	1.417749	1.449569

Differentials

The quantity x – c is the change in x, called Δx, and is called the **differential of x**, denoted by dx.
The change in y is $\Delta y = f(c + \Delta x) - f(c)$. Δy is *approximated* by the **differential of y**, denoted by dy: $dy = f'(x)\,dx$.

Δy is the *actual change* in y; dy is the *approximate change* in y.

<u>Example</u>: Compare Δy and dy for $y = 1 - 2x^2$ at x = 0. Δx = dx = -0.1.

$$\Delta y = f(c + \Delta x) - f(c) = [1 - 2(0 - 0.1)^2 - [1 - 2(0)^2] = -0.02$$

$$f'(x) = -4x\,dx \qquad dy = f'(x)\,dx = -4(0)(-0.1) = 0$$

The relationship between *derivatives* and *differentials* is that derivatives have the form dy/dx, while in differentials, dy and dx are on separate sides of the equation.

For example, for $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$

the derivative $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} = \frac{x-1}{2x\sqrt{x}}$

but the differential is: $dy = \left[\frac{x-1}{2x\sqrt{x}}\right]dx$

Propagated Error

If an error Δx is made in a measurement, then the true value is $x \pm \Delta x$. If another quantity is calculated from the measurement, the original measurement error creates another error in the calculated quantity. The error in f(x) is the propagated error, and is calculated by $dy = f'(x)\,dx$.

The **relative error** is $\pm\frac{dy}{y}$. The **percent error** is $\frac{dy}{y}(100)$, without the ± sign.

Example: the side of a square is measured to be 16 inches, with a possible error of 1/32 inches. Approximate the propagated error, the relative error, and the percent error in the computed area.

$A = x^2$ $\quad x = 16 \quad dx = \pm 1/32$

$dA = 2x\,dx = 2(16)(\pm 1/32) = \pm 1$ sq. in.

relative error $= \frac{dA}{A} = \frac{\pm 1}{16^2} = \pm 0.0039 \rightarrow$ 0.39% error

Example: The base and height of a triangle are measured to be 36 and 50 cm, with a possible error in each of .25 cm. Approximate the propagated error, relative error, and percent error in the computed area.

$A = \frac{1}{2}bh \quad b = 36 \quad h = 50 \quad db = dh = \pm .25$

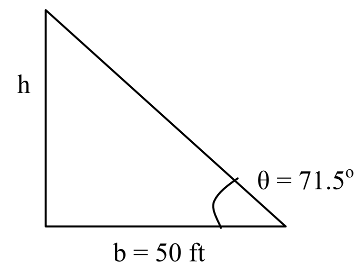
$dA = \frac{1}{2}(b\,dh + h\,db) = \frac{1}{2}[(36)(\pm .25) + (50)(\pm .25)] = \pm 10.75$ sq. cm.

relative error $= \frac{dA}{A} = \frac{\pm 10.75}{(36)(50)/2} = \pm 0.0119 \rightarrow$ 1.19% error

Example: A surveyor standing 50 ft from a tree sights the top of the tree at an elevation angle of 71.5^o. If the error in estimating the height of the tree is to be less than 6%, how accurate does the angle measurement have to be?

NOTE: When working with derivatives of trig functions, angles must be in radians!

$\tan\theta = h/b \rightarrow h = b\tan\theta$
$dh = b\sec^2\theta\, d\theta$



$71.5^o = 1.2479$ rad

$\tan(1.2479) = h/50 \rightarrow h = 50 * \tan(1.2479)$

$dh = 50 * \sec^2(1.2479)\, d\theta$

$$\left|\frac{dh}{h}\right| = \left|\frac{50\sec^2(1.2479)\,d\theta}{50\tan(1.2479)}\right| < 0.06$$

$$|d\theta| < (0.06)(\tan(1.2479))(\cos^2(1.2479))$$

$$|d\theta| < 0.018 \text{ rad} \quad \text{or} \quad |d\theta| < 1.03^o$$

Approximating function values

Function values can be approximated by using $f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx$

Example: Approximate the value of $(5.4)^2$

$f(x + \Delta x) \approx f(x) + f'(x)dx$

$f(x) = x^2 \qquad f'(x) = 2x\,dx \qquad x = 5 \qquad dx = \Delta x = .4$

$f(5 + .4) \approx 5^2 + 2(5)(.4) = 25 + 4 = 29 \qquad$ actual value: $(5.4)^2 = 29.16$

Example: Approximate the value of $\sqrt[4]{624}$.

$f(x) = \sqrt[4]{x} \qquad f'(x) = \frac{1}{4x^{3/4}}dx \qquad x = 625 \qquad dx = \Delta x = -1$

$$f(625 - 1) \approx \sqrt[4]{625} + \frac{1}{4(625)^{3/4}}(-1) = 5 - \frac{1}{500} = 4.998$$

actual value: $\sqrt[4]{624} = 4.998$

MAT 204 – Differentials and Increments

Suppose that z is a function of x and y, z = f(x, y) , and x and y change by small **increments Δx** and **Δy**. The **increment of z**, called **Δz,** is the *true* change in z, defined as:

$$\boldsymbol{\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)}$$

Example: $z = x^2 + y^2$, find Δz

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$
$$= (x + \Delta x)^2 + (y + \Delta y)^2 - (x^2 + y^2)$$
$$= 2x\Delta x + \Delta x^2 + 2y\Delta y + \Delta y^2$$

or $\Delta z = (2x + \Delta x)\Delta x + (2y + \Delta y)\Delta y$

If z = f(x, y), the **differentials** of x and y are called dx and dy.
The **total differential of z**, called **dz**, is the *approximate* change in z, defined as:

$$\boldsymbol{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

Example: $z = f(x, y) = x^2 + y^2$, find dz

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$dz = 2x\,dx + 2y\,dy$$

When Δx and Δy are small, we can substitute them for dx and dy to use the differential dz as an approximation to the increment Δz, that is: $dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y \approx \Delta z$.

Example: A rectangle whose length and width are 15" and 8" respectively, are measured with a defective tape that reads the dimensions as 14 7/8" and 8 1/4". What is the approximate increment in the area of the rectangle?

$$A = lw \qquad \frac{\partial A}{\partial l} = w \qquad \frac{\partial A}{\partial w} = l \qquad l = 15 \qquad w = 8 \qquad dl = -\frac{1}{8} \qquad dw = \frac{1}{4}$$

$$dA = \frac{\partial A}{\partial l} dl + \frac{\partial A}{\partial w} dw = w\,dl + l\,dw = 8(-1/8) + 15(1/4) = -1 + 15/4 \quad \rightarrow \quad \Delta A \approx \frac{11}{4}$$

Example: $z = \sqrt{x^2 + y^2}$

Approximate $z = \sqrt{(4.01)^2 + (3.2)^2}$ using the differential

$x = 4 \quad \Delta x = .01 \qquad y = 3 \quad \Delta y = .2$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$dz = \frac{x}{\sqrt{x^2 + y^2}} \Delta x + \frac{y}{\sqrt{x^2 + y^2}} \Delta y$$

$$dz = \frac{4}{5}(0.01) + \frac{3}{5}(0.2)$$

$dz = 0.1280 \quad \rightarrow \qquad z \approx 5 + 0.128 = 5.128$

the increment $\Delta z = \sqrt{(4.01)^2 + (3.2)^2} - \sqrt{4^2 + 3^2} = 0.1303$

Propagated Error

If $z = f(x, y)$ and x and y are the *true* values, whereas $x + \Delta x$ and $y + \Delta y$ are the *measured* values, then Δx is the measurement error in x and Δy is the measurement error in y.

Suppose the true values of the length and width of a rectangle are 15 and 8, but are measured to be 15.03 and 8.16. Then $x = 15$, $y = 8$, and the measurement errors are $\Delta x = .03$, $\Delta y = .16$.

The difference $\mathbf{f(x + \Delta x, y + \Delta y) - f(x, y)}$ is called the propagated error which is equal to $\mathbf{\Delta z}$.

To obtain an approximation to Δz, we use the total differential dz. For this particular rectangle :

$A = xy \qquad \partial A / \partial x = y \qquad \partial A / \partial y = x \qquad x = 15 \quad \Delta x = .03 \qquad y = 8 \quad \Delta y = .16$

and the propagated error is:

$$\begin{aligned} dA &= y\Delta x + x\Delta y \\ &= (8)(.03) + (15)(.16) \\ &= 2.6400 \end{aligned} \qquad (\Delta A = 2.6448)$$

The relative error in the area is the ratio of the propagated error to the correct area, which in this case, since $A = (15)(8) = 120$, is $dA/A = 2.64/120 = .022$. The percent error is (relative error)*100 = 2.2%.

Exercises

1. Find the total differential dz:
   a) $z = x^2 y$
   b) $z = x\sin y + y\cos x$
   c) $z = (x + y)/(x - y)$

2. Evaluate f(2, 3) and f(2.06, 3.2) and find Δz. Then find dz to approximate Δz.
   a) $z = \sqrt{y^2 - x^2}$
   b) $z = y\cos x$
   c) $z = x/(x + y)$

3. The radius r and height h of a right circular cone are measured with errors of 5% and 3%, respectively. What is the propagated error in the volume?

Answers

1. a. $dz = 2xydx + x^2dy$ b. $dz = (\sin y - y\sin x)dx + (x\cos y + \cos x)dy$

   c. $dz = \dfrac{-2ydx + 2xdy}{(x-y)^2}$

2. a. $\Delta z = \sqrt{3.2^2 - 2.06^2} - \sqrt{3^2 - 2^2} = 0.21269$ $dz = \dfrac{-xdx + ydy}{\sqrt{y^2 - x^2}} = \dfrac{-2(.06) + 3(.2)}{\sqrt{3^2 - 2^2}} = 0.2146$

   b. $\Delta z = 3.2\cos(2.06) - 3\cos(2) = -0.25531$

   $dz = -y\sin x dx + \cos x dy = -3(\sin 2)(.06) + (\cos 2)(.2) = -0.24690$

   c. $\Delta z = 2.06/(2.06+3.2) - 2/(2+3) = -0.00836$ $dz = \dfrac{ydx - xdy}{(x+y)^2} = \dfrac{3(.06) - 2(.2)}{(2+3)^2} = -0.0088$

3. $V = \dfrac{\pi r^2 h}{3}$ $dV = \dfrac{2\pi rh\,dr}{3} + \dfrac{\pi r^2\,dh}{3}$ $dr = \pm.05r$ $dh = \pm.03h$

   $dV = \dfrac{2\pi rh(\pm.05r)}{3} + \dfrac{\pi r^2(\pm.03h)}{3} = \dfrac{\pi r^2 h}{3}(\pm.13)$

   $\dfrac{dV}{V} = \pm.13 \rightarrow dV = \pm 0.13V$ (percent error of 13%)