CONTINUITY OF FUNCTIONS OF TWO VARIABLES

When we were considering the continuity of a function of one variable at a particular point, say x = a, recall that three conditions had to be met: f(a) had to be defined, $\lim_{x \to a} f(x)$ had to exist, and $\lim_{x \to a} f(x)$ had to be equal to *f(a)*. In the case of a function of two variables, for the function to be continuous at a particular point, the function has to be defined at the point and the limit has to exist at that point. In other words, if at the point (x_o, y_o):

$$f(x_0, y_0) = \lim_{(x,y) \to (x_0, y_0)} f(x,y), \text{ then } f(x,y) \text{ is continuous at } (x_0, y_0).$$

Example: Is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous at (0,0)?

Since $\lim_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist, the function is not continuous at (0,0).

If a function f(x,y) is not continuous at (x_o, y_o), but its limit exists, then by defining $f(x_o, y_o)$ equal to its limit, f(x,y) becomes continuous at (x_o, y_o). In such a case we say the discontinuity of f(x,y) at (x_o, y_o) is removable. We can do this with a function which has a discontinuity at a point only if the limit of the function exists at that point. We merely define the function to be equal to that limit. If the limit does not exist at that point, the discontinuity is not removable.

Example: The function $f(x,y) = \frac{5x^2 y}{x^2 + y^2}$ is not continuous at (0, 0). However, the limit at (0, 0) does exist and is equal to zero at that point. If we define f(x, y) to be equal to zero at (0, 0), then we have removed the point of discontinuity and the function becomes continuous at that point.

If f(x,y) and g(x,y) are two functions that are continuous at (x_o, y_o), then the function $\frac{f(x,y)}{g(x,y)}$ is also continuous at (x_o, y_o) only if $g(x_o, y_o) \neq 0$.

Example: $f(x,y) = 5x^2y$ is continuous at (0, 0)

$g(x,y) = x^2 + y^2$ is continuous at (0, 0)

However, $\frac{f(x,y)}{g(x,y)}$ is not continuous at (0, 0) because g(0, 0) = 0.

Problems: Determine whether the function is continuous at the stated point.

1. $f(x,y) = \frac{xy}{x^2 + y^2}$ at (1,0).
2. $f(x,y) = \frac{2x^2 - y^2}{x^2 + y^2}$ at (0,0)
3. $f(x,y,z) = \frac{2x - z}{x + y + z}$ at (0, 0, 0)
4. $f(x,y) = \begin{cases} \frac{(x^2 - y^2)}{(x^2 + y^2)} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ at (0, 0)
5. $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$ at (0, 0)

Answers:

1. Yes 2. No 3. No 4. No 5. Yes