NOTES – Infinite Series (Chpt 9)

9.1 Sequences $\{a_n\}$

If $\lim_{n\to\infty} a_n = L$, where L is any real number, the sequence converges, otherwise it diverges (goes to either ∞ or -∞).

<u>To find limits:</u>

1) "Divide out", as in:

$$\{a_n\} = \frac{n}{1-2n} \quad \to \quad \lim_{n\to\infty}\left(\frac{n}{1-2n}\right) = \lim_{n\to\infty}\left(\frac{1}{1/n-2}\right) = -\frac{1}{2} \quad \to \quad \text{converges}$$

2) If $f(n) = a_n$, use known limits of $f(x)$:

$$a_n = \left(1+\frac{1}{n}\right)^n \quad \to \quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \quad \to \quad \lim_{n\to\infty} a_n = e \quad \to \quad \text{converges}$$

Other handy limits: $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$ $\qquad \lim_{n\to\infty}\frac{x^n}{n!} = 0$ $\qquad \lim_{n\to\infty}\sqrt[n]{n} = 1$ $\qquad \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$

3) If $f(n) = a_n$, use L'Hôpital's Rule on $\lim_{x\to\infty} f(x)$:

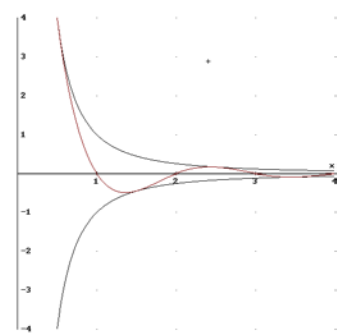
$$\{a_n\} = \frac{n^2}{2^n-1} \quad \to \quad \lim_{x\to\infty}\left(\frac{x^2}{2^x-1}\right) = \lim_{x\to\infty}\left(\frac{2x}{(\ln 2)2^x}\right) = \lim_{x\to\infty}\left(\frac{2}{(\ln 2)^2 2^x}\right) = 0 \to \text{converges}$$

4) Squeeze Theorem

if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ *and* $a_n \le c_n \le b_n$ then for all n > some integer N, $\lim_{n\to\infty} c_n = L$

This says that if you can fit a sequence between two other convergent sequences that have the same limit (or between a convergent sequence and a number), then the sequence in question will also have the same limit, and thereby converge. It doesn't have to fit between the sequences right away (at n = 1), as long as it's always squeezed after *some* n value.

<u>Example</u>: in the sequence $\frac{\sin n\pi}{n^2}$, the value of the numerator is $-1 \le \sin n\pi \le 1$, so the sequence can be written as between two sequences: $-\frac{1}{n^2} \le \frac{\sin n\pi}{n^2} \le \frac{1}{n^2}$



The limit of both $-\frac{1}{n^2}$ and $\frac{1}{n^2}$ is zero, so the limit of $\frac{\sin n\pi}{n^2}$ is also zero and the sequence converges.

Remember, the limits of sums, differences, products and quotients of sequences are the same as the sums, differences, products and quotients of individual limits, so some manipulation may be needed.

For example: $\lim_{n\to\infty}\left[\left(\frac{3n^2}{5n-1}\right)\sin\left(\frac{1}{n}\right)\right] = \lim_{n\to\infty}\left[\left(\frac{3n}{5n-1}\right)\left(n\sin\left(\frac{1}{n}\right)\right)\right] = \lim_{n\to\infty}\left(\frac{3n}{5n-1}\right)\lim_{n\to\infty}\left[n\sin\left(\frac{1}{n}\right)\right]$

$= \left(\frac{3}{5}\right)(1) = \frac{3}{5}$

If a sequence has an alternating sign in it, like $(-1)^n$, then test the limit of the absolute value:

if $\lim_{n\to\infty}|a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

A sequence $\{a_n\}$ is called **bounded** if it has *both* an upper bound that it cannot be greater than *and* a lower bound that it cannot be less than. A sequence is **monotonic** if its terms are all nondecreasing (increasing or equal) or all nonincreasing (decreasing or equal).

If $\{a_n\}$ is bounded and monotonic, it converges.

[This is not a test for divergence! A sequence can be not bounded or not monotonic and still converge.]

Note: limits at infinity are also horizontal asymptotes of the graph $f(n) = a_n$, so sometimes you can apply the asymptote rules to check your limits.

9.2 Series - sum of a sequence: $\sum_{n=1}^{\infty} a_n$

The partial sum of a series is found by adding a certain number of terms in the sequence.

If the sequence of partial sums $\{S_n\}$ has a finite limit (convergent), then the series $\sum_{n=1}^{\infty} a_n$ converges.

If the sequence of partial sums $\{S_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: A sequence can converge ($\lim_{n\to\infty} a_n = L$ *) but its series may diverge!*

A **telescoping series** is characterized by a sequence where many of the terms cancel each other, leaving a finite number of terms that determines the sum.

A telescoping series converges if a_n approaches a finite limit.

In the series: $(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots\ldots (a_n - a_{n+1})$ all the terms cancel except for the first one and the last one, and the n^{th} partial sum $S_n = a_1 - a_{n+1}$.

If this series converges, the sum $S = a_1 - \lim_{n\to\infty} a_{n+1}$.

Try creating a telescoping series using partial fractions:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots.\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

All the terms cancel except $\frac{1}{1} - \frac{1}{n+1}$. $\lim_{n\to\infty} \frac{1}{n(n+1)} = 0$, so this series converges (finite limit)

The sum is $S_n = a_1 - \lim_{n\to\infty} a_{n+1} = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1$.

Note: cancellation may not start right away, and not all of the terms toward the end may cancel - you may have more than one term left over at the beginning or the end of the series. Write out the series until you know the pattern.

Geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ and converges if $0 < |r| < 1$ diverges if $|r| \geq 1$

If you don't start n at zero: $\sum_{n=3}^{\infty} ar^n = \sum_{n=0}^{\infty} ar^n - (ar^0 + ar^1 + ar^2)$

repeating decimals: $.080\overline{808} = \sum_{n=0}^{\infty} \frac{8}{10^2}\left(\frac{1}{10^2}\right)^n = \frac{8/100}{1-1/100} = \frac{8}{99}$

nth term limit for convergent series: if a_n converges then $\lim_{n\to\infty} a_n = 0$

nth term test for divergence: if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Note: If $\lim_{n\to\infty} a_n = 0$, we know nothing! This is not a test for convergence of a series (except a telescoping series). In fact, the series could either converge or diverge.

9.3 Integral test: if f(x) *continuous, positive, decreasing*, $x \geq N$, and $a_n = f(n)$,

then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge

Remember, an improper integral converges if the limit exists for $\lim_{b\to\infty} \int_a^b f(x)\,dx$.

The infinite series does not necessarily converge to the limit of the integral.

<u>Example</u>**:** $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}} \rightarrow \int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}dx = \left[\operatorname{arcsec} x\right]_2^{\infty} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \rightarrow$ both converge

If the series converges to S, then the remainder $R_N = S - S_N$ is bounded by $0 \leq R_N \leq \int_N^{\infty} f(x)dx$.

<u>Example</u>: Find the max error when $\sum_{n=1}^{\infty} \frac{9}{n^3}$ is estimated using the first seven terms.

$R_7 \leq \int_7^{\infty} \frac{9}{x^3}dx \rightarrow R_7 \leq -\frac{9}{2x^2}\Big|_7^{\infty} = .0918$

Example: How many terms are needed to estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001?

$$R_N \le \int_N^{\infty} \frac{1}{x^3}dx \le .001 \quad \rightarrow \quad -\frac{1}{2x^2}\bigg|_N^{\infty} \le .001 \quad \rightarrow \quad \frac{1}{2N^2} \le .001 \quad \rightarrow \quad N \ge 23$$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if p >1; diverges if 0 < p ≤ 1

$\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series: 1 + 1/2 + 1/3 + 1/4 + …. (diverges)

For a series similar to the harmonic series, if the terms are larger than those in the harmonic series (e.g., 1/(n – 1)), expect it to diverge; if its terms are smaller (e.g., 1/(n ln n)), you can’t tell – try the integral test.

9.4 Comparisons of series

Direct comparison test: compare a series to another series whose convergence/divergence is known

for $0 < a_n \le b_n$ if $\sum b_n$ converges then $\sum a_n$ converges (b_n is the comparison series)
if $\sum a_n$ diverges then $\sum b_n$ diverges (a_n is the comparison series)

Example: $\frac{1}{\sqrt{n^3+3n}}$ looks like $\frac{1}{\sqrt{n^3}}$, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$ is a convergent p-series, and

$0 \le \frac{1}{\sqrt{n^3+3n}} \le \frac{1}{\sqrt{n^3}}$ so, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+3n}}$ also converges.

Example: $\frac{3^n}{2^n-1}$ looks like $\left(\frac{3}{2}\right)^n$, $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n$ is a divergent geometric series, and

$0 \le \left(\frac{3}{2}\right)^n \le \frac{3^n}{2^n-1}$ so, $\sum_{n=0}^{\infty}\frac{3^n}{2^n-1}$ also diverges.

Limit comparison test:

For a given $a_n > 0$ use a $b_n > 0$ with known convergence/divergence to compare it to.

If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ = L, finite and positive, the series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

To formulate the comparison series, ignore the coefficients and all but highest powers of n in the numerator and denominator of the original series.

Examples

a_n	b_n	$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$	
$\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	$\frac{1}{3}$	both converge
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{3}}$	both diverge
$\sum_{n=1}^{\infty} \frac{n^2-10}{4n^5+n^3}$	$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$	$\frac{1}{4}$	both converge

9.5 Alternating series: $a_n > 0$

$\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if $\lim_{n\to\infty} a_n = 0$ **and** $a_{n+1} \le a_n$

Approximating a sum

To approximate the sum of an alternating series, calculate the remainder after doing a partial sum:
$| R_n | \le a_{n+1}$.
For example, the remainder after summing the first 6 terms would be $| R_6 | \le a_7$

The total sum will be in the interval defined by: $S_n - | R_n | \le S \le S_n + | R_n |$

To find how many terms are needed to approximate a sum to within a certain error:
if desired error < .0001 → $a_{n+1} < .0001$ → find a_{n+1} by trial and error
the number of terms of the series needed will be n (if the series starts with n = 1)

Example: Approximate $\frac{1}{\sqrt{e}}$ with an error less than 0.001 using the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

by trial and error: $a_3 = 0.0208$ $a_4 = 0.0026$ $a_5 = 0.00026 < 0.001$

5 terms are needed to achieve the desired accuracy (note the series starts with n = 0):

$$\sum_{n=0}^{4} \frac{(-1)^n}{2^n n!} = 0.60677 \qquad \frac{1}{\sqrt{e}} = 0.60653 \qquad | R | = 0.00024$$

Absolute convergence: If $\sum |a_n|$ converges, then $\sum a_n$ converges

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges. This theorem can be used to test not only alternating series but also other series that involve sign changes that don't strictly alternate.

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$$ has positive and negative terms, but is not an alternating series

$$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2} \quad \rightarrow \quad \text{converges by Direct Comparison Test}$$

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$$ converges absolutely

If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is <u>conditionally convergent</u>

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ converges but } \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges, so } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \text{ is conditionally convergent}$$

9.6 Ratio Test:

$\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ $\qquad$ $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ (or $\lim = \infty$)

inconclusive if lim = 1

The ratio test is especially good to use for series with exponents and/or factorials.

<u>Example</u>: Does $\sum_{n=1}^{\infty}\frac{4^n}{n!}$ converge or diverge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!}\cdot\frac{n!}{4^n}\right| = \lim_{n\to\infty}\left|\frac{4}{(n+1)}\right| = 0 \quad \rightarrow \quad \text{converges}$$

Root Test:

$\sum a_n$ converges absolutely if $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$ $\qquad$ $\sum a_n$ diverges if lim > 1 (or $\lim = \infty$)

inconclusive if lim = 1

The root test also works well for series with n^{th} powers.

<u>Example</u>: Does $\sum_{n=1}^{\infty}\frac{n^{\sqrt{2}}}{11^n}$ converge or diverge?

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{n^{\sqrt{2}}}{11^n}} = \lim_{n\to\infty}\frac{\sqrt[n]{n^{\sqrt{2}}}}{11} = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^{\sqrt{2}}}{11} = \frac{1}{11} \quad \rightarrow \quad \text{converges}$$

Strategies

1. nth term approach zero? If not, diverges.
2. Special type – geometric, p-series, telescoping, alternating?
3. Apply Integral Test, Root Test, Ratio Test?
4. Compare the series to one of the special types?

9.7 Taylor polynomial functions ($P_n(x)$) approximate other elementary functions ($f(x)$) around a central value c. In other words, the polynomial will produce a similar graph to the function graph over a limited range around c. <u>f(x) must have n derivatives at c.</u>

for P_n expanded around a value c: $P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots \frac{f^n(c)}{n!}(x-c)^n$

$f(c) = P_0$ if c = 0 $\rightarrow$ Maclaurin polynomial

<u>Example</u>: Find $P_6(x)$ of cos(x) around center c = 0.

$f(x) = \cos x$ $f'''(x) = \sin x$ $f^{(6)}(x) = -\cos x$

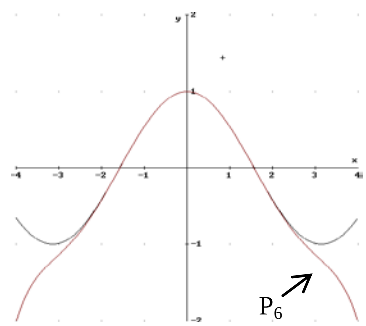
$f'(x) = -\sin x$ $f^{(4)}(x) = \cos x$

$f''(x) = -\cos x$ $f^{(5)}(x) = -\sin x$ $c = 0$

$$P_6(x) = \cos(0) - \sin(0)(x-0) + \frac{-\cos(0)}{2!}(x-0)^2 + \frac{\sin(0)}{3!}(x-0)^3$$

$$+ \frac{\cos(0)}{4!}(x-0)^4 + \frac{-\sin(0)}{5!}(x-0)^5 + \frac{-\cos(0)}{6!}(x-0)^6$$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$



To approximate values using either Taylor or Maclaurin:

cos (0.1): expand cos(x) around c = 0 $\rightarrow$ $P_6(x) = 1 - x^2/2! + x^4/4! - x^6/6!$

$x = 0.1$ $\rightarrow$ $P_6(0.1) = 0.995004165$ vs 0.995004165 actual

ln(1.1): expand ln(x) around c = 1 $\rightarrow$ $P_4(x) = (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4$

$x = 1.1$ $\rightarrow$ $P_4(1.1) = .095308333$ vs .095310180 actual

ln(1.1): expand ln(1 + x) around c = 0 $\rightarrow$ $P_4(x) = x - x^2/2 + x^3/3 - x^4/4$

$x = 0.1$ $\rightarrow$ same answer

Taylor's Theorem – estimating the remainder

Error (remainder) = $| R_n(x) | = | f(x) - P_n(x) |$

If f is differentiable through n + 1 in interval I containing c, for each x there is a <u>z between x and c</u>

such that $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ and $| R_n(x) | \leq \frac{|x-c|^{n+1}}{(n+1)!} \max| f^{(n+1)}(z) |$.

To find the $\max| f^{(n+1)}(z) |$, you can often use the endpoints of the interval between x and c.

Example: Find the error in approximating sin(0.1) with a 3^{rd} order approximation.

sin x (with c = 0) → $P_3(x) = x - x^3/3!$

$P_3(0.1) = 0.09983333$ c = 0 x = 0.1 0 < z < 0.1 $f^{(4)}(x) = \sin x$

$R_3(x) \le \frac{(.1)^4}{4!} \max|\sin z|$ and max | sin z | = sin 0.1

$R_3 \le .00000042$ (actual R = 0.09983342 – 0.09983333 = 0.00000009)

How many terms for desired accuracy?

Determine the degree of the Maclaurin polynomial P_n to use to approximate $e^{0.75}$ with an error < 0.001.

$f(x) = e^x$ $f^{(n+1)}(x) = e^x$ $x = 0.75$ $c = 0$ $0 < z < 0.75$ $\max|e^z| = e^{0.75}$

$$\frac{|x-c|^{n+1}}{(n+1)!}\max|f^{(n+1)}(z)| = \frac{e^{0.75}(0.75)^{(n+1)}}{(n+1)!} < 0.001$$ by trial and error, n = 5

9.8 Power Series: represent f(x) exactly (over a specified interval)

$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ power series centered at c

The domain of f(x) is all x for which the series converges (always includes c).
The domain can be either a point, an interval, or the entire real line.

Every power series converges at its center: $\sum_{n=0}^{\infty} a_n (c-c)^n = a_0$. **[for series, $(x - c)^0 = 1$, even if x = c]**

Radius of Convergence (R)

Only one is true:

1) R = 0 if the series converges only at c.
2) The series converges absolutely for **$|x - c| < R$** and diverges for $|x - c| > R$.
   R determines the interval of convergence around c: **(c – R, c + R).**
3) R = ∞ if the series converges absolutely for all x.

To find the radius of convergence, use the **Ratio Test**: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

1) $\sum_{n=0}^{\infty}\frac{(3n)!(x-1)^{2n}}{n!}$ c = 1 $\lim_{n\to\infty}\left|\frac{\frac{(3n+3)!(x-1)^{2n+2}}{(n+1)!}}{\frac{(3n)!(x-1)^{2n}}{n!}}\right| = \lim_{n\to\infty} 3(x-1)^2(3n+1)(3n+2) = \infty$

diverges for $|x - 1| > 0$

converges only at the center c = 1 → R = 0

2) $\sum_{n=0}^{\infty}(4x)^n$ c = 0 $\lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(4x)^n}\right| = 4|x|$

converges if $4|x| < 1$ or $|x| < ¼$ R = ¼ → converges on interval (-¼, ¼)

3) $\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}$ c = 0 $\lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{(n+1)!}}{\frac{(2x)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| = 0 < 1$

converges for all x → R = ∞

To find the <u>exact interval of convergence</u>, the endpoints need to be tested separately for convergence.

$\sum_{n=0}^{\infty}\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}$ c = 2

for |x| > 2, $\lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+2}}{(n+2)3^{n+2}}}{\frac{(x-2)^{n+1}}{(n+1)3^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(x-2)(n+1)}{3(n+2)}\right| = \frac{|x-2|}{3} < 1$

converges if $|x – 2| < 3$

R = 3 and the series converges on *at least* the interval (-1, 5).

Test the original series for the endpoints x = - 1 and x = 5:

If x = –1, the series is $\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{(n+1)3^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ which converges by the alternating series test.

If x = 5, the series is $\sum_{n=0}^{\infty}\frac{3^{n+1}}{(n+1)3^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ which is a divergent p-series.

The interval of convergence is [-1, 5).

<u>Differentiation and Integration of Power Series</u>

If $f(x) = \sum_{n=0}^{\infty}a_n(x-c)^n = a_o + a_1(x – c) + a_2(x – c)^2 + a_3(x – c)^3 + \ldots\ldots$ has R > 0,
then f(x) is differentiable on (c – R, c + R), and:

$f'(x) = \sum_{n=1}^{\infty}na_n(x-c)^{n-1} = a_1 + 2a_2(x – c) + 3a_3(x – c)^2 + \ldots..$ <u>same R</u>

$$\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{(n+1)} = C + a_o(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + \ldots \quad \underline{\text{same R}}$$

Note: The power rules of derivation and integration apply <u>only</u> to the exponent on the (x – c) term. Any other "n's" in the original series are not affected – they are part of a_n – e.g., $(-1)^n$.

Radii of convergence are the same. Intervals of convergence may differ because of endpoint behavior.

9.9 Representation of functions with power series

Geometric power series

A function can be represented by a geometric series if it has the form $\frac{a}{1-r}$, because $\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$.

The <u>radius of convergence</u> for a geometric series is: $\lim_{n\to\infty}\left|\frac{r^{n+1}}{r^n}\right| < 1$ or **| r | < 1**.

In $f(x) = \frac{1}{1-x}$, a = 1, r = x, and the function can be represented by $\sum_{n=0}^{\infty} x^n$ <u>on its interval of convergence,</u> which would be the radius $|x| < 1$ around the center c = 0, or (–1, 1).

The graph of $1 + x + x^2 + x^3 + \ldots$ will match the graph of $\frac{1}{1-x}$ on the interval (–1, 1).

To determine a geometric series **with center c**, where $f(x) = \frac{1}{1-x}$:

1) replace x in f(x) with (x – c) and <u>compensate for that subtraction</u>:

$c = -1 \quad \rightarrow$ replace x with (x + 1) $\rightarrow \quad \frac{1}{1-(x+1)+1} = \frac{1}{2-(x+1)}$

2) change to $\frac{a}{1-r}$: $\quad \frac{1}{2-(x+1)} = \frac{1/2}{1-(x+1)/2} \quad \rightarrow \quad a = ½$ and $r = (x + 1)/2$

$\frac{1}{1-x}$ centered at $-1 = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x+1}{2}\right)^n = ½\,[1 + (x+1)/2 + (x+1)^2/4 + (x+1)^3/8 + \ldots]$

This converges when $\left|\frac{x+1}{2}\right| < 1 \rightarrow |x+1| < 2 \qquad$ **(|x – c| < R)**

The interval of convergence is (-3, 1) and the graphs will match on that interval.

<u>Example</u>: $f(x) = \frac{3}{4-x}$; find the geometric power series centered at -2

$$\frac{3}{4-x} = \frac{3}{4-(x+2)+2} = \frac{3}{6-(x+2)} = \frac{1/2}{1-(x+2)/6} = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x+2}{6}\right)^n$$

converges when $\left|\frac{x+2}{6}\right| < 1 \quad \rightarrow \quad |x+2| < 6$ or on the interval (-8, 4)

Handy operations

1) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

This means you can build a series for x and then substitute x^N for x in the series.

Example: Find a power series for $f(x) = \frac{2}{2+x^2}$ centered at c = 0.

Find a series for $\frac{2}{2+x}$: $\quad \frac{2}{2+x} = \frac{1}{1-(-x/2)} = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^n$

Substitute x^2 for x: $\quad \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{2}\right)^n \qquad$ converges when $\left|\frac{x^2}{2}\right| < 1$

$\left|\frac{x^2}{2}\right| < 1 \rightarrow \left|x^2\right| < 2 \rightarrow -\sqrt{2} < x < \sqrt{2}$ or interval of convergence $\left(-\sqrt{2}, \sqrt{2}\right)$

2) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)\, x^n$

This is useful if you can split up a function into two functions, like using partial fractions.

Example: Find a power series for $f(x) = \frac{2x}{x^2-1}$ centered at c = 0.

$$\frac{2x}{x^2-1} = \frac{1}{x+1} + \frac{1}{x-1}$$

$$\frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \qquad |x| < 1$$

$$\frac{1}{x-1} = \frac{-1}{1-x} = \sum_{n=0}^{\infty}(-1)x^n \qquad |x| < 1$$

$$\frac{2x}{x^2-1} = \sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}(-1)x^n = \sum_{n=0}^{\infty}\left[(-1)^n - 1\right]x^n$$ interval of convergence (-1, 1)

Note: If you add two series with <u>*different*</u> *intervals of convergence, the interval of convergence of the sum will be the* <u>*intersection*</u> *of the separate intervals.*

9.10 Taylor and Maclaurin series

If a power series converges to f(x), then it must be a Taylor or Maclaurin series:

Taylor: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$ or Maclaurin c = 0: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ Remainder $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}$

If $\lim_{x\to\infty} R_n = 0$ for all x in interval I, then the Taylor series for f converges and equals f(x).

Binomial series

The series for $f(x) = (1+x)^k$ is $1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \ldots \frac{k(k-1)\ldots(k-n+1)x^n}{n!}$

R = 1 the series converges on (-1, 1)

$f(x) = \sqrt{1+x} = (1+x)^{½} \to 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1\cdot 3x^3}{2^3 \cdot 3!} - \frac{1\cdot 3\cdot 5x^4}{2^4 \cdot 4!} + \ldots\ldots$ R = 1

Using the table of Taylor series

Basic power series can be adapted to functions by:

substitution – can be done in the basic series.

Find the power series for e^{-2x}.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!} = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \ldots.$$

addition and subtraction can be done either with the basic series or with series in expanded form.

Find the series for $e^{ix} + e^{-ix}$.

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \ldots.$$

$$e^{-ix} = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} + \frac{(-ix)^6}{6!} + \ldots.$$

$$e^{ix} + e^{-ix} = 2 - x^2 + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \ldots = 2\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 2\cos x$$

or $$\sum_{n=0}^{\infty} \left[\frac{(ix)^n}{n!} + \frac{(-ix)^n}{n!} \right] = 2\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

Multiplying or dividing series must be done using the expanded forms, by long multiplication or division.

Approximating definite integrals: approximate $\int_0^{1/2} \frac{\ln(x+1)}{x}$ using a power series

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \qquad \frac{\ln(x+1)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}((x+1)-1)^n}{xn} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{n}$$

$$\int_0^{1/2} \frac{\ln(x+1)}{x} = \int_0^{1/2} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{n} \right] dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n^2} \Bigg|_0^{1/2}$$

for an approximation with an error < .0001, $|a_{n+1}|$ must be < .0001 (alternating series remainder)

$$\frac{(1/2)^{n+1}}{(n+1)^2} = \frac{1}{2^{n+1}(n+1)^2} < .0001 \rightarrow n+1=8 \rightarrow \text{ we need seven terms for the approximation}$$

$$x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \frac{x^6}{36} + \frac{x^7}{49} \Bigg|_0^{1/2} = .448458$$

actual integral is .448414, error is .000044.

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
***n*th-Term**	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n\to\infty} b_n = L$		Sum: $S = b_1 - L$
***p*-Series**	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$		Remainder: $\lvert R_N\rvert \leq a_{N+1}$
Integral (*f* is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$
Direct Comparison (a_n, b_n > 0)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison (a_n, b_n > 0)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

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(recreated from page 644 in *Calculus 8th edition* by Larson, Hostetler, Edwards)

Power Series for Elementary Functions

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n (x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!\, x^{2n+1}}{(2^n n!)^2 (2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$ Convergence at x = ±1 depends on the value of *k*.

NOTE: The binomial series is valid for noninteger values of *k*. Moreover, if *k* happens to be a positive integer, the binomial series reduces to a simple binomial expansion.

Powers and multiples of *x* can be directly substituted for *x* in the series. Series can be added, subtracted, and directly multiplied or divided by powers and multiples of *x*. All of this can be performed on the n^{th} term of the series. However, two different series must be multiplied or divided together like polynomials; the n^{th} term cannot be used. A few of these concepts are demonstrated on the back.

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Example 1: Find the power series for $\cos\sqrt{x}$.

Using the power series for $\cos x$, you can replace x by $\sqrt{x}$ to obtain the series

$$\cos\sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots$$

This series converges for $x \geq 0$.

Example 2: Find the first three nonzero terms for $e^x \arctan x$.

Using the Maclaurin series for e^x and $\arctan x$ in the table, you have

$$e^x \arctan x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)$$

Multiply these expressions and collect like terms as you would for multiplying polynomials. So,

$$e^x \arctan x = x + x^2 + \tfrac{1}{6}x^3 + \cdots$$

Example 3: Find the first three nonzero terms for $\tan x$.

Using the Maclaurin series for $\sin x$ and $\cos x$ in the table, you have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

Divide using long division. So, $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Example 4: Find the power series for $e^x + e^{-x}$.

Since we are adding two series, we can add the n^{th} terms together.

$$e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \frac{2}{0!} + 0 + \frac{2x^2}{2!} + 0 + \frac{2x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{2(x^{2n})}{(2n)!}$$