

Notes – Vector Analysis (Chpt 15)

15.1 Vector Fields

vector field in a plane: $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$

vector field in space: $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

gradient $\nabla f = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ is a vector field

inverse square field:

given position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

vector field \mathbf{F} is an inverse square field if:

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2} \mathbf{u} \quad k = \text{real number} \quad \mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \text{ (unit vector)}$$

sketching vector fields – can sometimes sketch by finding vectors of equal magnitude

example: $\mathbf{F}(x, y) = -x\mathbf{i} + y\mathbf{j}$

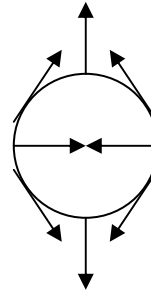
if $\|\mathbf{F}\| = c$, then $\sqrt{x^2 + y^2} = c$, or $x^2 + y^2 = c^2$

→ vectors of equal magnitude radiate from circles of radius c

for $c = 1 = \|\mathbf{F}\|$:

| (x, y) | \mathbf{F} |
|--|---|
| $(1, 0)$ | $-\mathbf{i}$ |
| $(0, 1)$ | \mathbf{j} |
| $(-1, 0)$ | \mathbf{i} |
| $(0, -1)$ | $-\mathbf{j}$ |
| $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ | $-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ |

etc.



example: $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

$4x^2 + y^2 = c^2$ → vectors of equal magnitude radiate outward from vertically aligned ellipses

example: $\mathbf{v}(x, y, z) = (36 - x^2 - y^2)\mathbf{k}$

$x^2 + y^2 = 36 - c^2$ → vectors of equal magnitude radiate in the \mathbf{k} direction from circles in the xy plane, with the longer vectors nearer the origin

conservative vector fields

sometimes, vectors are normal to the level curve from which they radiate, like gradients

In the plane, $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is conservative iff. $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

If \mathbf{F} is conservative, then $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$; f is the potential function for \mathbf{F}

Every inverse square field is conservative.

Finding the potential function (recovering a function from its gradient): integrate

$$1. \mathbf{F} = 2xy\mathbf{i} + (x^2 - y)\mathbf{j} = \nabla f \quad \frac{\partial N}{\partial x} = 2x \quad \frac{\partial M}{\partial y} = 2x \rightarrow \text{conservative}$$

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} \quad f_x = 2xy \quad f_y = x^2 - y$$

$$\int (2xy)dx = x^2y + g(y) \quad \int (x^2 - y)dy = x^2y - y^2/2 + h(x)$$

$$g(y) = -y^2/2 \quad h(x) \text{ does not show up in } f_x, \text{ so must} = \text{constant } K$$

$$\rightarrow \mathbf{f} = x^2y - y^2/2 + K$$

$$2. \mathbf{F} = 2xy^3\mathbf{i} + 3y^2x^2\mathbf{j} = \nabla f \quad \frac{\partial N}{\partial x} = 6xy^2 \quad \frac{\partial M}{\partial y} = 6xy^2 \rightarrow \text{conservative}$$

$$f_x = 2xy^3 \quad f_y = 3y^2x^2$$

$$\int (2xy^3)dx = x^2y^3 + g(y) \quad \int (3y^2x^2)dy = x^2y^3 + h(x) \quad g(y) = h(x) = K$$

$$\rightarrow \mathbf{f} = x^2y^3 + K$$

Curl of a vector field in space

The curl of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is $\nabla \times \mathbf{F}(x, y, z) =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

If $\nabla \times \mathbf{F}(x, y, z) = 0$, then \mathbf{F} is irrotational and conservative, and a potential function f can be found

Example: find a potential function for $\mathbf{F}(x, y, z) = 3x^2y^2z\mathbf{i} + 2x^3yz\mathbf{j} + x^3y^2z\mathbf{k}$

$$\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} = 2x^3y - 2x^3y = 0 \quad \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} = 3x^2y^2 - 3x^2y^2 = 0$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6x^2yz - 6x^2yz = 0 \rightarrow \text{conservative}$$

$$f_x = 3x^2y^2z \quad f_y = 2x^3yz \quad f_z = x^3y^2$$

$$\int 3x^2y^2z dx = x^3y^2z + g(y, z) \quad \int 2x^3yz dy = x^3y^2z + h(x, z)$$

$$\int x^3y^2 dz = x^3y^2z + k(x, y) \quad \rightarrow \mathbf{f} = x^3y^2z + K$$

Divergence of a vector field

$$\text{Divergence } \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \quad \text{or} \quad \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

If $\nabla \cdot \mathbf{F} = 0$, then the field is divergence-free

If M, N, and P have continuous second partial derivatives, then $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

15.2 Line Integrals – integrating a function over a path

$\int_C f(x, y, z) ds$ = the line integral of f along a piecewise smooth curve C of finite length.

For C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

$$ds = \|\mathbf{r}'(t)\| dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt$$

$$\text{and } \int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt$$

$$\text{if } f(x, y, z) = 1 \quad \text{then } \int_C 1 ds = \int_a^b \|\mathbf{r}'(t)\| dt = \text{length of } C$$

Example: evaluate $\int_C (x + y^2 + 4z) ds$ over the straight line from (0, 3, 2) to (1, 2, 5)

$$\begin{aligned} \text{Parameterize the line:} \quad & x = t & x' &= 1 & 0 \leq t \leq 1 \\ & y = 3 - t & y' &= -1 \\ & z = 2 + 3t & z' &= 3 \end{aligned}$$

$$ds = \sqrt{1^2 + (-1)^2 + 3^2} dt = \sqrt{11} dt$$

$$\int_C (x + y^2 + 4z) ds = \int_0^1 (t + (3-t)^2 + 4(2+3t)) \sqrt{11} dt = \sqrt{11} \int_0^1 t^2 + 7t + 17 dt$$

$$= \sqrt{11} \left[\frac{t^3}{3} + \frac{7t^2}{2} + 17t \right]_0^1 \approx 69.096$$

Example: compute the mass of a spring in two turns ($0 \leq t \leq 4\pi$) of a circular helix given by $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2t \mathbf{k}$ where density $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{dr} = -3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} + 2 \mathbf{k} \quad ds = \sqrt{9 \sin^2 t + 9 \cos^2 t + 4} dt = \sqrt{13} dt$$

mass = $\int \rho ds$ from \mathbf{r} , $x = 3 \cos t$, $y = 3 \sin t$, and $z = 2t \rightarrow$ substitute into ρ

$$\int_C \rho ds = \int_C \frac{1}{2}(x^2 + y^2 + z^2) ds = \int_0^{4\pi} \frac{1}{2}(9 \cos^2 t + 9 \sin^2 t + 4t^2) \sqrt{13} dt$$

$$= \frac{\sqrt{13}}{2} \int_0^{4\pi} (9 + 4t^2) dt = \frac{2\sqrt{13}\pi}{3} (27 + 64\pi^2)$$

Line integrals of vector fields

Can be used to find work done in a force field \mathbf{F}

- consider only the part of \mathbf{F} acting in the direction of motion of the object
- projection of \mathbf{F} onto unit tangent vector \mathbf{T} : $\mathbf{F} \cdot \mathbf{T}$
- force in direction of motion is $(\mathbf{F} \cdot \mathbf{T})\mathbf{T}$
- motion in opposite directions along the same path results in opposite signs of the work done
- work is positive in the direction of the force field
- work is negative in the direction against the force field

In a conservative vector field (like inverse square fields), the work done in moving an object from one point to another is independent of path.

$$\text{work } \mathbf{W} = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) ds$$

$$\mathbf{F} \cdot \mathbf{T} ds = \mathbf{F} \cdot \mathbf{r}'(t) dt = \mathbf{F} \cdot d\mathbf{r}$$

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt$$

From $\mathbf{r}(t)$, find $x(t)$, $y(t)$, $z(t)$ and substitute into \mathbf{F}

Example: find the work done by the force $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ on a particle moving on the path $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t^2 \mathbf{k}$ from $(0, 1, 0)$ to $(1, 0, \pi^2/4)$.

From \mathbf{r} : $x(t) = \sin t$ $y(t) = \cos t$ $z(t) = t^2$
 since $z = t^2 \rightarrow 0 \leq t \leq \pi/2$

Substitute into \mathbf{F} : $\mathbf{F}(t) = \sin^2 t \mathbf{i} + \cos^2 t \mathbf{j} + t^4 \mathbf{k}$

$$\mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + 2t \mathbf{k}$$

$$\int_a^b \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{\pi/2} \sin^2 t \cos t + \cos^2 t \sin t + 2t^5 dt = \frac{\sin^3 t}{3} + \frac{\cos^3 t}{3} + \frac{t^6}{3} \Big|_0^{\pi/2} = \frac{\pi^6}{192}$$

Example: find the work done by a person weighing 180 lb climbing one revolution (2π) of a circular helical staircase if the person rises 10 feet vertically.

The equation for a helix is $\mathbf{r}(t) = R\sin t \mathbf{i} + R\cos t \mathbf{j} + ct \mathbf{k}$
 where R = radius of the helix and $2\pi c$ = vertical separation of the loops

$$\begin{aligned}\mathbf{r}(t) &= 3\sin t \mathbf{i} + 3\cos t \mathbf{j} + (10/2\pi)t \mathbf{k} & 0 \leq t \leq 2\pi \\ \mathbf{r}'(t) &= 3\cos t \mathbf{i} - 3\sin t \mathbf{j} + (10/2\pi) \mathbf{k} \\ \mathbf{F} &= 180\mathbf{k}\end{aligned}$$

$$\int_a^b \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{2\pi} \frac{1800}{2\pi} dt = \frac{1800}{2\pi} t \Big|_0^{2\pi} = 1800 \text{ ft}\cdot\text{lb}$$

Note that work done in a gravitational field, which is conservative, is independent of path, and is the same as if the person had risen directly vertically ($= 180 \text{ lb} \cdot 10 \text{ ft}$).

Differential form: $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$$\int_c \mathbf{F} \cdot \mathbf{r}' dt = \int_a^b \left[M \frac{dx}{dt} + N \frac{dy}{dt} \right] dt = \int_a^b M dx + N dy$$

Note that M and N come from \mathbf{F} and dx and dy come from \mathbf{r}'

Example: evaluate $\int_c (y - 3x)dx + x^2 dy$ along the path $\mathbf{r}(t) = 2t\mathbf{i} + 10t\mathbf{j}$ $0 \leq t \leq 1$

from \mathbf{r} : $x = 2t$, $dx = 2dt$ $y = 10t$, $dy = 10dt$

substitute into \mathbf{F} :

$$\int_0^1 [(10t - 6t)(2) + 4t^2(10)] dt = \int_0^1 8t + 40t^2 dt = \frac{52}{3}$$

or, *without converting to t :*

\mathbf{r} is a straight line going from $(0, 0)$ to $(2, 10) \rightarrow x = x$, $y = 5x$; $dx = dx$, $dy = 5dx$

$$\int_c (y - 3x)dx + x^2 dy = \int_0^2 (5x - 3x)dx + 5x(5dx) = \int_0^2 2x + 5x^2 dx = x^2 + \frac{5}{3}x^3 \Big|_0^2 = \frac{52}{3}$$

Example: *without converting to t*

Evaluate $\int_c (2x - y)dx + (x + 3y)dy$ along the path $y = x^3$ from $(0, 0)$ to $(2, 8)$

$$x = x, dx = dx; y = x^3, dy = 3x^2 dx \rightarrow \int_0^2 (2x - x^3)dx + (x + 3x^3)(3x^2 dx)$$

$$= \int_0^2 2x + 2x^3 + 9x^5 dx = 108$$

or, by converting to t :

$$x = t, y = t^3; dx = dt, dy = 3t^2 dt \quad (\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}) \quad 0 \leq t \leq 2$$

$$\begin{aligned} \int_c (2x - y)dx + (x + 3y)dy &= \int_0^2 (2t - t^3)(dt) + (t + 3t^2)(3t^2 dt) \\ &= \int_0^2 2t + 2t^3 + 9t^5 dt = 108 \end{aligned}$$

Note the integrals and limits of integration are the same whether using x or t .

Example: in force field $\mathbf{F}(x, y) = 15[(4 - x^2y)\mathbf{i} - xy\mathbf{j}]$, find the value of c for motion along the parabolic path $y = c(1 - x^2)$ from $(-1, 0)$ to $(1, 0)$ that results in the least amount of work done.

This problem could be worked by parameterizing the path with $x = t$ and $y = c(1 - t^2)$ and $-1 \leq t \leq 1$, but it makes no difference to the integral or the limits of integration

Using the differential form: $x = x, dx = dx; y = c - cx^2, dy = -2cxdx$

$$\begin{aligned} W &= \int_a^b M dx + N dy = \int_{-1}^1 15[4 - x^2(c - cx^2)]dx - 15[x(c - cx^2)(-2cxdx)] \\ &= \int_{-1}^1 60 - 15x^2(c - 2c^2) + 15x^4(c - 2c^2) dx = 60x - 5x^3(c - 2c^2) + 3x^5(c - 2c^2) \Big|_{-1}^1 \\ &= 120 - 4c + 8c^2 \end{aligned}$$

$$W' = -4 + 16c = 0 \text{ when } c = 1/4 \rightarrow W = 120 - 4(1/4) + 8(1/4)^2 = 119.5$$

What if we move in a straight line path from $(-1, 0)$ to $(1, 0)$?

$x = x, y = 0; dx = dx, dy = 0; \mathbf{F} = 60\mathbf{i}$ on that path

$$W = \int_a^b M dx + N dy = \int_{-1}^1 60 dx = 120$$

Obviously, this is not a conservative field:

$$\frac{\partial N}{\partial x} = -15y \quad \frac{\partial M}{\partial y} = -15x^2 \quad \frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$$

15.3 Conservative Vector Fields and Independence of Path

Fundamental theorem of line integrals

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative and M and N are continuous in the region, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is the potential function of \mathbf{F} ($\mathbf{F}(x, y) = \nabla f(x, y)$)

An equation \mathbf{r} for the curve is not needed in a conservative field.

K 's are not necessary in the potential function f , since they are cancelled by subtraction

Example: $\mathbf{F}(x, y) = 2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where \mathbf{r} is a piecewise smooth curve from $(-1, 1)$ to $(3, 2)$. \mathbf{F} is conservative.

$$\int 2(x + y) dx = x^2 + 2xy + g(y) \quad \int 2(x + y) dy = 2xy + y^2 + h(x)$$
$$f = x^2 + 2xy + y^2 = (x + y)^2$$

$$f(3, 2) - f(-1, 1) = 5^2 - 0 = 25$$

Example: $\mathbf{F}(x, y, z) = 6x\mathbf{i} - 4z\mathbf{j} - (4y - 20z)\mathbf{k}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where \mathbf{r} is a piecewise smooth curve from $(0, 0, 0)$ to $(3, 4, 0)$. \mathbf{F} is conservative.

$$\int 6x dx = 3x^2 + g(y, z) \quad \int -4z dy = -4yz + h(x, z) \quad \int 20z - 4y dz = 10z^2 - 4yz + k(x, y)$$

$$f = 3x^2 - 4yz + 10z^2 \quad f(3, 4, 0) - f(0, 0, 0) = 27$$

evaluating a line integral using equivalent conditions

if $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ has continuous 1st partial derivatives in an open connected region and C is a piecewise smooth curve, then the following conditions are equivalent:

1. \mathbf{F} is conservative ($\mathbf{F} = \nabla f$)
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve

Example:

evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = e^z y \mathbf{i} + e^z x \mathbf{j} + e^z xy \mathbf{k}$ and $\mathbf{r} = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3 \mathbf{k} \quad 0 \leq t \leq \pi$

you could proceed as normal with $d\mathbf{r}$ and substituting functions of t for x , y , and z :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi -16e^3 \sin^2 t + 16e^3 \cos^2 t + 16e^3 \cos t \sin t \, dt$$

or: test \mathbf{F} and find that it's conservative, then find f :

$$\int e^z y \, dx = e^z xy + g(y, z) \quad \int e^z x \, dy = e^z xy + h(x, z) \quad \int e^z xy \, dz = e^z xy + k(x, y)$$

$$\rightarrow f = e^z xy + K$$

The initial and end points for \mathbf{r} are $(4, 0, 3)$ and $(-4, 0, 3)$

Since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, we can evaluate f :

$$e^z xy \Big|_{(4,0,3)}^{(-4,0,3)} = 0$$

Conservation of energy

potential energy p of a particle in a conservative vector field \mathbf{F} : $p(x, y, z) = -f(x, y, z)$

work done by \mathbf{F} moving a particle from point A to B is:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(x, y, z) \Big|_A^B = -p(x, y, z) \Big|_A^B = p(A) - p(B)$$

Work = difference in potential energies of A and B; sum of potential and kinetic energies remains constant.

15.4 Green's Theorem

A curve C is simple if it does not cross itself.

A region R is simply connected if its boundary consists of one simple curve.

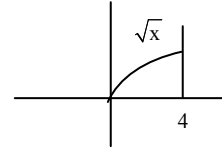
Let region R be a simply connected region with a piecewise smooth boundary C oriented counterclockwise (region R is always to the left of C). If M and N have continuous 1st partial

derivatives, then: $\int_C M dx + N dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

where dA is $dx dy$, $dy dx$, $r dr d\theta$, etc.

The line integral over a closed curve in a conservative field is still zero $\left(\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \right)$

Example: Evaluate the line integral $\int_C y^2 dx + xy dy$ where C is the boundary of the region lying between $y = 0$, $y = \sqrt{x}$, and $x = 4$.



$$\int_C y^2 dx + xy dy = \int_0^4 \int_0^{\sqrt{x}} (y - 2y) dy dx = \int_0^4 -\frac{x}{2} dx = -4$$

Example: calculating work

Calculate the work done on a particle moving along the same path as above while subject to the force $\mathbf{F} = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$.

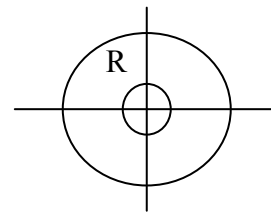
$$\int_C (3x^2 + y)dx + 4xy^2 dy = \int_R \int (4y^2 - 1) dA = \text{work}$$

$$\int_0^4 \int_0^{\sqrt{x}} (4y^2 - 1) dy dx = \int_0^4 \left(\frac{4}{3} x^{\frac{3}{2}} - \sqrt{x} \right) dx = \left(\frac{8}{15} x^{\frac{5}{2}} - \frac{2}{3} x^{\frac{3}{2}} \right) \Big|_0^4 = \frac{176}{15}$$

Finding a line integral by area

If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$, then $\int_R \int 1 dA = \text{area of region R}$

Example: Evaluate $\int_C (y - x) dx + (2x - y) dy$ where C encloses the region inside the circle $x^2 + y^2 = 16$ and outside the circle $x^2 + y^2 = 1$.



$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$$

$$\int_R \int 1 dA = 16\pi - \pi = 15\pi$$

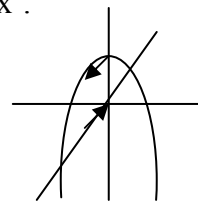
Finding area by a line integral

One choice for $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$ is $M = -y/2$ and $N = x/2$.

$$\text{Then } A = \frac{1}{2} \int_C x dy - y dx$$

Example: find the area bounded by the curves $y = 2x + 1$ and $y = 4 - x^2$.

C1: $y = 2x + 1 \quad dy = 2dx$
 $\frac{1}{2} \int_C xdy - ydx = \frac{1}{2} \int_{-3}^1 2xdx - (2x+1)dx = \frac{1}{2} \int_{-3}^1 -dx$



C2: $y = 4 - x^2 \quad dy = -2xdx$
 $\frac{1}{2} \int_C xdy - ydx = \frac{1}{2} \int_1^{-3} -2x^2 dx - (4-x^2)dx = \frac{1}{2} \int_1^{-3} (-x^2 - 4)dx$

$$\frac{1}{2} \int_{-3}^1 -dx + \frac{1}{2} \int_1^{-3} (-x^2 - 4)dx = \frac{32}{3}$$

Example: Find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$

$x = 3 \cos t \quad dx = -3 \sin t \quad y = 5 \sin t \quad dy = 5 \cos t \quad 0 \leq t \leq 2\pi$

$$A = \frac{1}{2} \int_C xdy - ydx = \frac{1}{2} \int_0^{2\pi} 15 \cos^2 t + 15 \sin^2 t dt = \frac{1}{2} \int_0^{2\pi} 15 dt = \frac{15t}{2} \Big|_0^{2\pi} = 15\pi$$

Centroid of a region with area A:

$$\bar{x} = \frac{1}{2A} \int_C x^2 dy \quad \bar{y} = -\frac{1}{2A} \int_C y^2 dx$$

Area of a plane region bounded by simple closed path C given in **polar coordinates**:

$$A = \frac{1}{2} \int_C r^2 d\theta$$

Alternative forms:

For $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + O\mathbf{k}$: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA$

For $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ and $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ where s is the arc length parameter:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla F) dA$$

15.5 Parametric Surfaces

For surfaces, vector-valued function is a function of two parameters, u and v .

Surface: $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$
Equations for the surface: $x = x(u, v)$ $y = y(u, v)$ $z = z(u, v)$

Example: identify the surface given by $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}$ $0 \leq v \leq 2\pi$

$$x^2 + y^2 = u^2 = z \rightarrow \text{paraboloid around the } z \text{ axis}$$

Example: identify the surface given by $\mathbf{r}(u, v) = 5 \cos v \cos u \mathbf{i} + 5 \cos v \sin u \mathbf{j} + 5 \sin v \mathbf{k}$

$$x^2 + y^2 = 25 \cos^2 v \quad x^2 + y^2 + z^2 = 25 \rightarrow \text{sphere with radius } 5$$

Finding equations for surfaces

if $z = f(x, y)$, then the surface is represented by $\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k}$

$$\text{for cylinder } z = x^2 + 4y^2 - 16: \quad \mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + (x^2 + 4y^2 - 16) \mathbf{k}$$

for a surface of revolution:

if revolve $y = f(x)$ around x -axis, $a \leq x \leq b$, use:

$$x = u \quad y = f(u) \cos v \quad z = f(u) \sin v \quad a \leq u \leq b \quad 0 \leq v \leq 2\pi$$

if revolve $x = f(z)$ about z -axis: $z = u \quad x = f(u) \cos v \quad y = f(u) \sin v$

for $y = f(x) = \sqrt{x}$ revolved around the x -axis:

$$x = u \quad y = \sqrt{u} \cos v \quad z = \sqrt{u} \sin v \\ \rightarrow \text{paraboloid of the form } x = y^2 + z^2$$

Normal vectors and tangent planes of a surface

If surface S is $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$

then at (x_0, y_0, z_0) the normal vector $\mathbf{N} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$

using $\mathbf{N} = \langle a, b, c \rangle$ the equation of the tangent plane is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example: find the equation of the tangent plane at the point $(2, 0, -1)$ to the elliptic paraboloid given by $\mathbf{r} = u \mathbf{i} + v \mathbf{j} - (u^2/4 + v^2) \mathbf{k}$

at $(2, 0, -1)$, $u = x = 2$ and $v = y = 0$

$$\mathbf{r}_u = \mathbf{i} - (u/2) \mathbf{k} = \mathbf{i} - \mathbf{k} \quad \mathbf{r}_v = \mathbf{j} - 2v \mathbf{k} = \mathbf{j} \quad \mathbf{r}_u \times \mathbf{r}_v = \mathbf{N}(2, 0) = \mathbf{i} + \mathbf{k} = \langle 1, 0, 1 \rangle$$

equation of the tangent plane is: $(x - 2) + (z + 1) = 0$ or $x + z = 1$

Example: find the equation of the tangent plane at the point $(-2, 0, 4)$ to the surface given by $\mathbf{r} = u \cosh v \mathbf{i} + u \sinh v \mathbf{j} + u^2 \mathbf{k}$

using the function \mathbf{r} and the point $(-2, 0, 4)$:

$$z = u^2 = 4 \rightarrow u = \pm 2$$

$$x = u \cosh v = -2 \rightarrow \mathbf{u} = -2, \cosh v = 1, \mathbf{v} = \mathbf{0} \quad (\cosh v \text{ cannot} = -1)$$

$$y = u \sinh v = 0 \quad \text{and} \quad \sinh(0) = 0 \rightarrow \text{checks}$$

$$\mathbf{r}_u = \cosh v \mathbf{i} + \sinh v \mathbf{j} + 2u \mathbf{k} = \mathbf{i} - 4\mathbf{k} \quad \mathbf{r}_v = u \sinh v \mathbf{i} + u \cosh v \mathbf{j} = -2\mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{N}(-2, 0) = -8\mathbf{i} - 2\mathbf{k} \quad \text{or} \quad \langle -8, 0, -2 \rangle$$

equation of the tangent plane is: $-8(x + 2) - 2(z - 4) = 0$ or $4x + z = -4$

Area of a parametric surface

For surface $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$ over open region D in uv -plane:

$$\text{surface area} = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dA$$

if $z = f(x, y)$ and $\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + f(x, y) \mathbf{k}$ then over region R in the xy -plane:

$$\text{surface area} = \iint_R \|\mathbf{r}_x \times \mathbf{r}_y\| dA = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA$$

Example: find the surface area of the paraboloid $\mathbf{r} = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + u^2 \mathbf{k}$ over the domain $0 \leq u \leq 2$ and $0 \leq v \leq 2\pi$

$$\mathbf{r}_u = 2\cos v \mathbf{i} + 2\sin v \mathbf{j} + 2u \mathbf{k} \quad \mathbf{r}_v = -2u \sin v \mathbf{i} + 2u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = -4u^2 \cos v \mathbf{i} - 4u^2 \sin v \mathbf{j} + 4u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 4u \sqrt{u^2 + 1}$$

$$\text{surface area} = \int_0^{2\pi} \int_0^2 4u \sqrt{u^2 + 1} du dv = \frac{8\pi}{3} (5\sqrt{5} - 1)$$

Example: find the surface area of the surface given by $\mathbf{r} = u \mathbf{i} + \sin u \cos v \mathbf{j} + \sin u \sin v \mathbf{k}$ over the domain $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$

$$\mathbf{r}_u = \mathbf{i} + \cos u \cos v \mathbf{j} + \cos u \sin v \mathbf{k} \quad \mathbf{r}_v = -\sin u \sin v \mathbf{j} + \sin u \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \sin u \cos u \mathbf{i} - \sin u \cos v \mathbf{j} - \sin u \sin v \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u \sqrt{\cos^2 u + 1}$$

$$\int_0^{2\pi} \int_0^{\pi} \sin u \sqrt{\cos^2 u + 1} \, du \, dv = \pi \left[2\sqrt{2} + \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \right]$$

15.6 Surface integrals

If S is a surface with equation $z = g(x, y)$ and R is its projection onto the xy -plane, then the surface integral of f over S is:

$$\iint_S f(x, y, z) \, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} \, dA$$

If $y = g(x, z)$ and R is in the xz -plane, then:

$$\iint_S f(x, y, z) \, dS = \iint_R f(x, g(x, z), z) \sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2} \, dA$$

If $x = g(y, z)$ and R is in the yz -plane, then:

$$\iint_S f(x, y, z) \, dS = \iint_R f(g(y, z), y, z) \sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2} \, dA$$

Example: evaluate the surface integral $\iint_S (x - 2y + z) \, dS$

where S is bounded by $z = 10 - 2x + 2y$, $0 \leq x \leq 2$, and $0 \leq y \leq 4$

using a projection onto the xy -plane: $g(x, y) = 10 - 2x + 2y$ $g_x = -2$ $g_y = 2$
 $\sqrt{1 + (-2)^2 + 2^2} = 3$

$$\iint_S (x - 2y + z) \, dS = \int_0^2 \int_0^4 (10 - x)(3) \, dy \, dx = 216$$

The integral can also be evaluated by solving the z equation for x or y and projecting onto the yx -plane or the xz -plane, respectively.

The mass of a surface area of variable density $\rho(x, y, z) = \iint_S \rho(x, y, z) dS$

Example: A hemispheric lamina S is given by $z = \sqrt{25 - x^2 - y^2} = g(x, y)$, $0 \leq z \leq 5$. If the density is $\rho = kz$, find the mass of the lamina.

Projection onto xy -plane: $x^2 + y^2 \leq 25$

$$g_x = \frac{-x}{\sqrt{25 - x^2 - y^2}} \quad g_y = \frac{-y}{\sqrt{25 - x^2 - y^2}} \quad \sqrt{1 + [g_x]^2 + [g_y]^2} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$\iint_S \rho(x, y, z) dS = \iint_S kz dS = \iint_R k \sqrt{25 - x^2 - y^2} \frac{5}{\sqrt{25 - x^2 - y^2}} dA = 5k \iint_R dA$$

changing to polar coordinates: $0 \leq r \leq 5$, $0 \leq \theta \leq 2\pi$

$$5k \iint_R dA = 5k \int_0^{2\pi} \int_0^5 r dr d\theta = 125k\pi$$

Parametric surfaces and surface integrals

For a surface given by: $\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$, the surface integral of $f(x, y, z)$ over S is:

$$\iint_S f(x, y, z) dS = \iint_D f(x(u, v), y(u, v), z(u, v)) \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\| dA$$

Example: evaluate the surface integral $\iint_S (x + y) dS$ for the surface S defined by:

$$\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + u \sin v \mathbf{j} + u \mathbf{k} \quad 0 \leq u \leq 4 \quad 0 \leq v \leq \pi$$

$$\mathbf{r}_u = 2 \cos v \mathbf{i} + \sin v \mathbf{j} + \mathbf{k} \quad \mathbf{r}_v = -2u \sin v \mathbf{i} + 2u \cos v \mathbf{j} \quad \|\mathbf{r}_u \times \mathbf{r}_v\| = 2u\sqrt{5}$$

$$\begin{aligned} \iint_S (x + y) dS &= \int_0^\pi \int_0^4 (2u \cos v + 2u \sin v) (2u\sqrt{5}) du dv = 4\sqrt{5} \int_0^\pi \int_0^4 u^2 (\cos v + \sin v) du dv \\ &= \frac{512\sqrt{5}}{3} \end{aligned}$$

Orientation of a surface

An orientable surface S has two distinct sides (a Möbius strip is not orientable). To orient it, choose one of two possible unit normal vectors.

For S given by $z = g(x, y)$, let $G(x, y, z) = z - g(x, y)$

S can be oriented by either

$$\mathbf{N} = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} = \frac{-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \quad (\text{"upward"})$$

or

$$\mathbf{N} = \frac{-\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} = \frac{g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} - \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \quad (\text{"downward"})$$

if surface S is: $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, then

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \quad \text{or} \quad \mathbf{N} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}$$

Flux integrals

The flux integral of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across surface S is $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$

$$= \iint_R \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}] \, dA \quad \text{or} \quad \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

Example: Let S be the 1st octant part of the sphere $z = g(x, y) = \sqrt{16 - x^2 - y^2}$. The normal vector is outward. A fluid with density ρ is flowing through the surface according to the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the rate of mass flow through the surface.

$$g_x = \frac{-x}{\sqrt{16 - x^2 - y^2}} \quad g_y = \frac{-y}{\sqrt{16 - x^2 - y^2}}$$

$$\begin{aligned} \iint_S \rho \mathbf{F} \cdot \mathbf{N} \, dS &= \rho \iint_R (x\mathbf{i} + y\mathbf{j} + \sqrt{16 - x^2 - y^2}\mathbf{k}) \cdot \left(\frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k} \right) \, dA \\ &= \rho \iint_R \frac{16}{\sqrt{16 - x^2 - y^2}} \, dA = 16\rho \int_0^{\pi/2} \int_0^4 \frac{1}{\sqrt{16 - r^2}} \, r \, dr \, d\theta = 32\pi\rho \end{aligned}$$

15.7 Divergence Theorem

Q is a solid region bounded by closed surface S with unit normal vector outward from Q. If \mathbf{F} is a vector field with continuous partial derivatives in Q, then:

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \operatorname{div} \mathbf{F} \, dV \quad \operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

So, instead of evaluating $\mathbf{F} \cdot \mathbf{N}$ for several surfaces, you only need one triple integral.

Example: Q is a solid region bounded by $0 \leq x \leq a$, $0 \leq y \leq a$, and $0 \leq z \leq a$ (a cube).

If $\mathbf{F} = x^2z \mathbf{i} - y \mathbf{j} + xyz \mathbf{k}$, find $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$.

$$\operatorname{div} \mathbf{F} = 2xz + xy - 1 \quad \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^a \int_0^a \int_0^a (2xz + xy - 1) \, dz \, dy \, dx = \frac{3a^5}{4} - a^3$$

Example: Q is bounded by the cone $z = +\sqrt{x^2 + y^2}$ and the plane $z = 4$.

$\mathbf{F} = (xy^2 + \cos z) \mathbf{i} + (x^2y + \sin z) \mathbf{j} + e^z \mathbf{k}$. Find $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS$.

$$\operatorname{div} \mathbf{F} = y^2 + x^2 + e^z$$

change to cylindrical coordinates: $r \leq z \leq 4$ $0 \leq r \leq 4$ $0 \leq \theta \leq 2\pi$ $x^2 + y^2 = r^2$

$$\iiint_Q (y^2 + x^2 + e^z) \, dV = \int_0^{2\pi} \int_0^4 \int_r^4 (r^2 + e^z) r \, dz \, dr \, d\theta = 2\pi(5e^4 + 251/5)$$

Since $\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_Q \operatorname{div} \mathbf{F} \, dV$ represents the total flux of \mathbf{F} into or out of Q through S,

a point in the vector field is a: source if $\operatorname{div} \mathbf{F} > 0$

sink if $\operatorname{div} \mathbf{F} < 0$

incompressible if $\operatorname{div} \mathbf{F} = 0$

Example: Q is the sphere $x^2 + y^2 + z^2 = 4$. Find the outward flux of $\mathbf{F} = xy \mathbf{i} + 4y \mathbf{j} + xz \mathbf{k}$ through the sphere.

$$\operatorname{div} \mathbf{F} = x + y + 4$$

spherical coordinates: $x = \rho \sin \phi \cos \theta$ $y = \rho \sin \phi \sin \theta$

$$\iiint_Q \operatorname{div} \mathbf{F} \, dV = \int_0^2 \int_0^\pi \int_0^{2\pi} (\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta + 4) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho = \frac{128\pi}{3}$$

15.8 Stoke's Theorem

If oriented surface S is bounded by a piecewise smooth simple closed curve C , then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\mathbf{curl} \mathbf{F}) \cdot \mathbf{N} dS = \iint_R \mathbf{curl} \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}] dA$$

Example: find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (3x - z)\mathbf{i} + (2x + 2z)\mathbf{j} + (2y + x)\mathbf{k}$ and C is the circle $x^2 + y^2 = 9$.

$$\mathbf{curl} \mathbf{F} = -2\mathbf{j} + 2\mathbf{k} \quad -g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k} = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$\iint_R \mathbf{curl} \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}] dA = \iint_R (4y + 2) dA$$

In polar coordinates: $y = r \sin\theta$ $0 \leq \theta \leq 2\pi$ $0 \leq r \leq 3$

$$2 \int_0^{2\pi} \int_0^3 (2r \sin\theta + 1) r dr d\theta = 2 \int_0^{2\pi} \int_0^3 (2r^2 \sin\theta + r) dr d\theta = 9 \int_0^{2\pi} (81 \sin\theta + 1) d\theta = 18\pi$$