

## NOTES – Functions of Several Variables (Chpt 13)

### 13.1 Functions of several variables $z = f(x, y)$ or $w = f(x, y, z)$

can add, subtract, multiply, divide, but cannot form composite functions

can only do composite if  $g$  is a function of a single variable and  $h$  is a function of several variables

then composite  $(g \circ h)(x, y) = g(h(x, y))$

eg.,  $f(x, y) = \sqrt{16 - 4x^2 - y^2}$  can be written as  $g(h(x, y))$

where  $g(u) = \sqrt{u}$  and  $h(x, y) = 16 - 4x^2 - y^2$

polynomial function of 2 variables: sum of functions  $cx^m y^n$ , eg,  $3xy^2 + x - 2$

rational function: quotient of 2 polynomial functions

graph of  $z = f(x, y)$  is a surface in space; the domain is the projection onto xy plane

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}$$

domain:  $4x^2 + y^2 \leq 16$  or  $\frac{x^2}{4} + \frac{y^2}{16} \leq 1 \rightarrow$  on or inside an ellipse in  $xy$  plane

range:  $0 \leq z \leq \sqrt{16}$  or  $0 \leq z \leq 4$

graph: upper half of an ellipsoid

**level curves:** scalar field  $z = f(x, y) = c$ , e.g., isotherms, contour maps

level curves of  $f(x, y) = x^2 + y^2 = c$  are circles

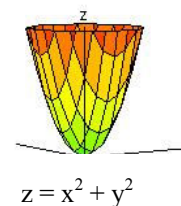
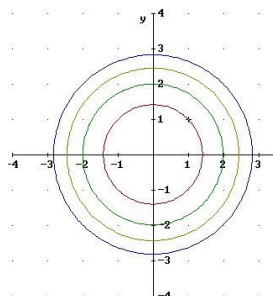
$$c = 0 \quad \text{radius} = 0$$

$$c = 2 \quad \text{radius} = \sqrt{2}$$

$$c = 4 \quad \text{radius} = 2$$

$$c = 6 \quad \text{radius} = \sqrt{6}$$

$$c = 8 \quad \text{radius} = 2\sqrt{2}$$

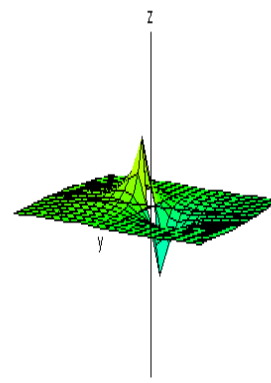
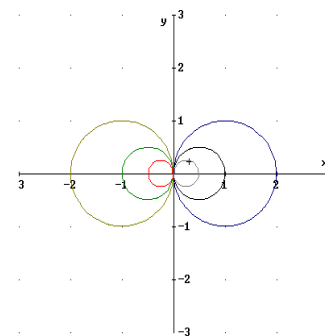


level curves of  $f(x, y) = \frac{x}{x^2 + y^2}$

$$\frac{x}{x^2 + y^2} = c \quad \rightarrow \quad x^2 + y^2 = \frac{x}{c}$$

$$x^2 - \frac{x}{c} + \frac{1}{4c^2} + y^2 = \frac{1}{4c^2}$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \frac{1}{4c^2}$$



these are circles with centers at  $\left(\frac{1}{2c}, 0\right)$  and radii of  $\frac{1}{2c}$

**level surfaces:** 3-D surfaces of  $f(x, y, z) = c$

level surfaces of  $f(x, y, z) = x^2 + y^2 - z = c$  are of the form  $z = x^2 + y^2 - c$   
→ nested elliptical (circular) paraboloids with “points” at  $-c$  on the  $z$ -axis

### 13.4 Differentials

**Increments:**  $\Delta x, \Delta y, \Delta z$       **Differentials:**  $dx, dy, dz$

if  $z = f(x, y)$ , then  $dx = \Delta x, dy = \Delta y$  and

the total differential  $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$  (etc for more variables)

$z = f(x, y)$  is differentiable at  $(x_0, y_0)$  if  $\Delta z$  can be written as:

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \text{ where } \varepsilon_1, \varepsilon_2 \rightarrow 0 \text{ as } (\Delta x, \Delta y) \rightarrow (0, 0)$$
$$[\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)]$$

Example       $f(x, y) = 2x^2 - 4y + 3$        $f_x = 4x$        $f_y = -4$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = 2(x + \Delta x)^2 - 4(y + \Delta y) + 3 - 2x^2 + 4y - 3$$
$$= (4x)\Delta x - (4)\Delta y + (2\Delta x)\Delta x + (0)\Delta y$$

$$\varepsilon_1 = 2\Delta x \quad \varepsilon_2 = 0 \quad \varepsilon_1, \varepsilon_2 \rightarrow 0 \text{ as } (\Delta x, \Delta y) \rightarrow (0, 0)$$

→  $f(x, y)$  is differentiable

if  $f_x$  and  $f_y$  are continuous in open region  $R$ , then  $f$  is differentiable on  $R$

#### Approximation by differentials

$dz \approx \Delta z$       linear approximation: approximate  $\Delta z$  by  $dz$

Example:  $z = xy$       approximate  $\Delta z$  as  $(x, y)$  moves from  $(1, 2)$  to  $(1.05, 2.1)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = ydx + xdy = y\Delta x + x\Delta y = 2(.05) + 1(.1) = .2$$

$$\text{actual } \Delta z = f(1.05, 2.1) - f(1, 2) = .205$$

#### Error analysis

Example:  $z = -1.83x - 1.09y + 28.7$        $x = 7.2 \pm .25$        $y = 8.5 \pm .25$

$$dz = -1.83dx - 1.09dy = -1.83\Delta x - 1.09\Delta y = -1.83(\pm .25) - 1.09(\pm .25) = \pm 0.73$$

$$z = -1.83(7.2) - 1.09(8.5) + 28.7 = 6.259$$

$$\text{relative error} = (0.73)/(6.259) = 11.7\%$$

Example  $a = \frac{v^2}{r}$  measurement errors of 2% and 1%:  $dv = \pm .02v$   $dr = \pm .01r$

$$da = \frac{\partial a}{\partial v} dv + \frac{\partial a}{\partial r} dr = \frac{2v}{r} dv - \frac{v^2}{r^2} dr = \frac{2v}{r} (\pm .02v) - \frac{v^2}{r^2} (\pm .01r) = \pm \frac{.04v^2}{r} \pm \frac{.01v^2}{r}$$

$$\text{max error (da)} = \pm \frac{.05v^2}{r} \quad \text{max relative error } \frac{da}{a} = 5\%$$

### 13.5 Chain rule

if  $w = f(x, y)$ ,  $x = g(t)$ , and  $y = h(t)$ , then:  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$  (one independent variable)

if  $w = f(x, y)$ ,  $x = g(s, t)$ , and  $y = h(s, t)$ , then:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad (\text{two independent variables})$$

can do substitution *before or after*:

Example Find  $dw/dt$  for  $w = \ln(x^2 + y^2)$   $x = 2t + 3$   $y = 4 - t$

$$\begin{aligned} 1) \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \frac{2x}{x^2 + y^2} (2) + \frac{2y}{x^2 + y^2} (-1) \\ &= \frac{4(2t+3)}{(2t+3)^2 + (4-t)^2} - \frac{2(4-t)}{(2t+3)^2 + (4-t)^2} = \frac{10t+4}{5t^2 + 4t + 25} \end{aligned}$$

or 2)  $w = \ln((2t+3)^2 + (4-t)^2) = \ln(5t^2 + 4t + 25)$

$$\frac{dw}{dt} = \frac{10t+4}{5t^2 + 4t + 25}$$

### Implicit partial differentiation

if  $F(x, y) = 0$  and defines  $y$  implicitly, then  $\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$

if  $F(x, y, z) = 0$  and defines  $z$  implicitly, then  $\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}$  and  $\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$

example  $F = x^2 + y^2 + z^2 + 6xw - 8w^2 - 5 = 0$  defines  $w$  as function of  $x, y, z$

$$F_x = 2x + 6w \quad F_y = 2y \quad F_z = 2z \quad F_w = 6x - 16w$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{2x + 6w}{6x - 16w} = -\frac{x + 3w}{3x - 8w} \quad \frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{2y}{6x - 16w} = -\frac{y}{3x - 8w}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{2z}{6x - 16w} = -\frac{z}{3x - 8w}$$

### 13.6 Directional derivative $D_u f(x, y)$ : slope at a point on a surface

surface  $z = f(x, y)$  point  $P(x_0, y_0)$  is in domain

slope at a point on surface in x-direction given by  $f_x$

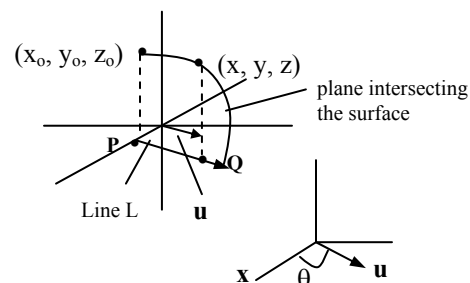
slope in y-direction given by  $f_y$

direction of  $D_u$ : **unit vector**  $\mathbf{u} = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}$

$Q(x, y)$  lies on line  $L$  in  $xy$  plane, where

$$x = x_0 + t \cos\theta \quad y = y_0 + t \sin\theta$$

points on a curve on the surface lie above line  $L$



$$D_u f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos\theta, y + t \sin\theta) - f(x, y)}{t}$$

if  $f$  is differentiable, then  $D_u = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$

infinite number of  $D_u$ 's at a point, depending on direction specified by  $\mathbf{u}$

if  $\theta = 0$  (x direction),  $D_u = f_x(x, y)$

if  $\theta = \pi/2$  (y direction),  $D_u = f_y(x, y)$

Example: For  $f(x, y) = x^2 - y^2$  and  $\mathbf{u} = \cos(\pi/4) \mathbf{i} + \sin(\pi/4) \mathbf{j}$ , find  $D_u$  on the surface above the point  $(4, 3)$

$$D_u = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

$$D_u = 2x \cos(\pi/4) - 2y \sin(\pi/4)$$

$$= 2(4) \left( \frac{\sqrt{2}}{2} \right) - 2(3) \left( \frac{\sqrt{2}}{2} \right)$$

$$= \sqrt{2}$$

#### gradient - del f or grad f

**vector in the xy plane:**  $\nabla$  is normal to the *level curve*

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \quad (\mathbf{u} \text{ is a unit vector})$$

if  $\nabla f = 0$ , then  $D_u = 0$

direction of max increase =  $\nabla f(x, y)$ ; max value of  $D_u = \|\nabla f\|$

direction of min increase =  $-\nabla f(x, y)$ ; min value of  $D_u = \|\nabla f\|$

if  $f(x, y)$  denotes a point (altitude) on a mountain,  $-\nabla f(x, y)$  is the *compass direction* (vector in the  $xy$  plane) to the path of steepest *descent*, or fastest rate of change direction on a topographical map of level curves

Example: On a mountain modeled by  $h(x, y) = 4000 - 0.001x^2 - 0.004y^2$ , which way is the path of steepest *ascent* when you are at the point  $(500, 300, 3390)$ ?

$$\nabla h(x, y) = h_x(x, y) \mathbf{i} + h_y(x, y) \mathbf{j} = -0.002x \mathbf{i} - 0.008y \mathbf{j} = -\mathbf{i} - 2.4 \mathbf{j} \quad (202.6^\circ \text{ bearing})$$

$$\|\nabla h\| = 2.6 \text{ ft/ft}$$

For  $f(x, y, z)$ :

$$D_u f(x, y, z) = af_x + bf_y + cf_z \quad \mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$$

$\nabla$  is normal to the *level surface*

Example: Find the path of a heat-seeking particle in a temperature field given by  $T(x, y) = 50 - x^2 - 2y^2$ , starting from the point (4, 3).

The path of maximum temperature increase can be represented by  $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ .

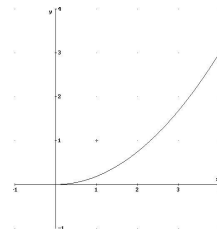
The tangent vector to the path at  $(x(t), y(t))$  is  $\mathbf{r}'(t) = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j}$ .

The tangent vector is in the same direction as the gradient  $\nabla T(x, y) = -2x\mathbf{i} - 4y\mathbf{j}$ .

$$-2x = k \frac{dx}{dt} \quad -4y = k \frac{dy}{dt} \quad \rightarrow \quad \frac{dx}{-2x} = \frac{dy}{-4y}$$

the solution is  $x = C\sqrt{y}$

$$\text{at } (4, 3), 4 = C\sqrt{3} \quad \text{and } C = \frac{4}{\sqrt{3}} \quad \rightarrow \quad \text{the path is } x = \frac{4}{\sqrt{3}}\sqrt{y}$$



### 13.7 tangent plane to surface S at P is a plane through P $\perp$ $\nabla F(P)$

for surface S given by  $z = f(x, y)$ , then  $F(x, y, z) = f(x, y) - z$   
and  $F(x, y, z) = 0$  (alternative equation of surface S)

S is the *level surface* of F given by  $F(x, y, z) = 0$

at a point  $P(x_0, y_0, z_0)$  on surface S given by  $F(x, y, z) = 0$  and  $\nabla F \neq \mathbf{0}$

- the plane through P *normal* to  $\nabla F(x_0, y_0, z_0)$  is the tangent plane to S at P

- the line through P *in direction of*  $\nabla F(x_0, y_0, z_0)$  is the normal line to S at P

#### equation of tangent plane:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

if  $z = f(x, y)$ , then  $F(x, y, z) = f(x, y) - z$ , and the equation of the tangent plane is

$$f_x(x - x_0) + f_y(y - y_0) - (z - z_0) = 0$$

for  $f(x, y)$ , if  $\nabla f(x, y) = \mathbf{0}\mathbf{i} + \mathbf{0}\mathbf{j}$  then the tangent plane is horizontal

Example. Find the point on the surface given by  $z = 3x^2 - 2xy + y^2 - 8y$  at which the tangent plane is horizontal.

We're concerned only with where  $\nabla f(x, y) = \mathbf{0}\mathbf{i} + \mathbf{0}\mathbf{j}$ .

$$\nabla f(x, y) = (6x - 2y)\mathbf{i} + (-2x + 2y - 8)\mathbf{j} = \mathbf{0}$$

$$6x - 2y = 0$$

$$\underline{-2x + 2y = 8}$$

$$x = 2, y = 6, z = -24$$

So the point is (2, 6, -24)

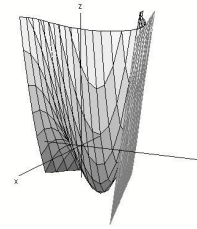
Example: Find the equation of the tangent plane to the surface given by  $z = x^3 - 3xy + y^3$  at the point (1, 2, 3).

$$F(x, y, z) = x^3 - 3xy + y^3 - z = 0$$

$$(3x_0^2 - 3y_0)(x - x_0) + (-3x_0 + 3y_0^2)(y - y_0) - (z - z_0) = 0$$

$$\text{at } (1, 2, 3): -3(x - 1) + 9(y - 2) - z + 3 = 0$$

$$\rightarrow 3x - 9y + z = -12$$



**equation of the normal line:**

find  $\nabla F = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$  and plug in  $P(x_0, y_0, z_0)$  values for  $x, y,$  and  $z$   
 or use  $\nabla F = a \mathbf{i} + b \mathbf{j} + c \mathbf{k}$  and make the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$\nabla f(x, y)$  is a vector in the  $xy$ -plane;  $\nabla F(x, y, z)$  is a vector in space

Example: Find the equation of the normal line to the surface  $x^2 + y^2 - z^2 = 0$  at the point (5, 12, 13)

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} = 10\mathbf{i} + 24\mathbf{j} - 26\mathbf{k}$$

$$\text{or } \frac{x-5}{10} = \frac{y-12}{24} = \frac{z-13}{-26}$$

**tangent line to the curve of intersection of two surfaces**

$$\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

Example: Find the equation of the tangent line to the intersection of the surfaces

$$z = \sqrt{x^2 + y^2} \quad \text{and} \quad 2x + y + 2z = 20 \quad \text{at the point } (3, 4, 5).$$

$$F = \sqrt{x^2 + y^2} - z = 0 \qquad G = 2x + y + 2z - 20 = 0$$

$$\nabla F = \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} - \mathbf{k} = (3/5)\mathbf{i} + (4/5)\mathbf{j} - \mathbf{k} \qquad \nabla G = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 2 & 1 & 2 \end{vmatrix} = (1/5)(13\mathbf{i} - 16\mathbf{j} - 5\mathbf{k})$$

**angle of inclination**

$\nabla F$  is normal to the tangent plane and  $\mathbf{k}$  is normal to the  $xy$ -plane.

The angle of inclination between the tangent plane and the  $xy$ -plane ( $0 \leq \theta \leq \pi/2$ ) is:

$$\cos \theta = \frac{|\nabla F \cdot \mathbf{k}|}{\|\nabla F\|}$$

Example: Find the angle of inclination of the tangent plane to the surface  $xy - z^2 = 0$  at the point  $(2, 2, 2)$

$$\nabla F = y\mathbf{i} + x\mathbf{j} - 2z\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

$$\frac{|\nabla F \cdot \mathbf{k}|}{\|\nabla F\|} = \frac{4}{\sqrt{24}} = \frac{2}{\sqrt{6}} \quad \arccos\left(\frac{2}{\sqrt{6}}\right) \approx 35.3^\circ$$

### 13.8 Extrema of functions

on an open region  $R$  the critical points are where:

- 1)  $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$   
or 2)  $f_x(x_0, y_0)$  or  $f_y(x_0, y_0)$  does not exist

relative extrema only at critical points

test for min or max:

if  $f(x, y) \geq f(x_0, y_0) \rightarrow$  relative minimum at  $(x_0, y_0)$

if  $f(x, y) \leq f(x_0, y_0) \rightarrow$  relative maximum at  $(x_0, y_0)$

Example: Determine the relative extrema for  $f(x, y) = -x^2 - 5y^2 + 8x - 10y - 13$

$$f_x = -2x + 8 \quad f_y = -10y - 10 \rightarrow \text{critical point at } (4, -1)$$

$$f(4, -1) = 8$$

by completing the squares,  $f(x, y) = -(x - 4)^2 - 5(y + 1)^2 + 8 \leq 8 \rightarrow$  relative maximum

Example: Determine the relative extrema for  $f(x, y) = \sqrt{x^2 + y^2} + 1$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \quad f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

partial derivatives do not exist at  $(0, 0)$ , and this is the only critical point

$$f(0, 0) = 1 \quad f(x, y) = \sqrt{x^2 + y^2} + 1 \geq 1 \rightarrow \text{relative minimum}$$

**2<sup>nd</sup> partials test** – find saddle points as well as min's and max's

Cannot use if either 1<sup>st</sup> derivative does not exist

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad d = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

1) if  $d > 0$  and  $f_{xx}(a, b) > 0 \rightarrow$  rel min at  $(a, b)$

2) if  $d > 0$  and  $f_{xx}(a, b) < 0 \rightarrow$  rel max at  $(a, b)$

3) if  $d < 0$  then  $(a, b, f(a, b))$  is a saddle point

4) if  $d = 0$  inconclusive, test fails

(**Note**:  $f_{xy}$  means do  $\delta/\delta x$  first, then  $\delta/\delta y$ , and means the same as  $\frac{\delta^2 f}{\delta y \delta x}$ .)

A relative extremum can also be an absolute extremum. Or, an absolute extremum can occur at a boundary point in the domain.

Example: Find the absolute extrema of  $f(x, y) = 2x - 2xy + y^2$  over the region R in the xy-plane bounded by  $y = x^2$  and  $y = 1$ .  $-1 \leq x \leq 1$

$$\begin{aligned} f_x &= 2 - 2y = 0 && \rightarrow y = 1 \\ f_y &= -2x + 2y = -2x + 2 && \rightarrow x = 1 \end{aligned}$$

at  $(1, 1)$ ,  $z = 1$ , but the 2<sup>nd</sup> partials test fails to categorize it as a min, max, or saddle point

over  $y = 1$ ,  $z = 2x - 2x + 1 = 1$

over  $y = x^2$ ,  $f(x, y) = f(x) = 2x - 2x^3 + x^4$   
 $f'(x) = 2 - 6x^2 + 4x^3$ , with critical values at  $x = 1$  and  $x = -\frac{1}{2}$

at  $(-\frac{1}{2}, \frac{1}{4})$ ,  $z = -11/16 \rightarrow$  absolute minimum  
 absolute maximum is on  $y = 1$  (including  $(1, 1)$ ),  $z = 1$

### 13.9 Applications

Example Find the minimum distance to the plane  $2x + 3y + z = 12$  from the point  $(1, 2, 3)$ .

Point on the plane:  $(x, y, 12 - 2x - 3y)$

Distance formula:  $D = d^2 = (x - 1)^2 + (y - 2)^2 + (9 - 2x - 3y)^2$

$$\begin{aligned} D_x &= 2(x - 1) - 4(9 - 2x - 3y) = 5x + 6y - 19 = 0 \\ D_y &= 2(y - 2) - 6(9 - 2x - 3y) = 6x + 10y - 29 = 0 \end{aligned}$$

$$\begin{aligned} 5x + 6y &= 19 \\ 6x + 10y &= 29 \end{aligned}$$

$$\frac{7x = 8}{\phantom{7x = 8}} \quad \rightarrow \quad x = 8/7, \quad y = 31/14, \quad z = 43/14 \quad \rightarrow \quad d = \frac{1}{\sqrt{14}}$$

Example Find the minimum distance to the paraboloid  $z = x^2 + y^2$  from the point  $(5, 0, 0)$ .

Point on the paraboloid:  $(x, y, x^2 + y^2)$

Distance formula:  $D = d^2 = (x - 5)^2 + y^2 + (x^2 + y^2)^2$

$$\begin{aligned} D_x &= 2(x - 5) + 4x(x^2 + y^2) = 4x^3 + 2x + 4xy^2 - 10 = 0 \\ D_y &= 2y + 4y(x^2 + y^2) = 4y^3 + 2y + 4x^2y = 0 \end{aligned}$$

To clear the xy terms, multiply  $D_x$  by  $x$  and  $D_y$  by  $-y$ :

$$\begin{aligned} 4x^4 + 2x^2 + 4x^2y^2 - 10x &= 0 \\ -4y^4 - 2y^2 - 4x^2y^2 &= 0 \\ \hline x(2x^3 + x - 5) - y^2(2y^2 + 1) &= 0 \end{aligned} \quad \rightarrow \text{ solutions are } (0, 0) \text{ and } (\sim 1.235, 0)$$

for  $(0, 0)$ ,  $d = 5$       for  $(1.235, 0)$ ,  $d = \sim 4.06 \rightarrow$  minimum

### Example

A rectangular box rests on the  $xy$ -plane with one vertex at the origin and the opposite vertex in the plane  $3x + 2y + z = 6$ . Find the dimensions that give the maximum volume.

$$x, y, z > 0$$

$$V = xyz = xy(6 - 3x - 2y) = 6xy - 3x^2y - 2xy^2$$

$$V_x = 6y - 6xy - 2y^2 = 0 \quad \text{and} \quad V_y = 6x - 3x^2 - 4xy = 0$$

solve by factoring:  $2y(3 - 3x - 2y) = 0$  and  $x(6 - 3x - 4y) = 0$

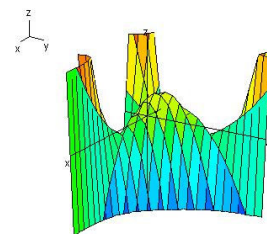
$$\text{yields critical points } (2/3, 1), (0, 0), (0, 3), \text{ and } (2, 0)$$

(exclude the ones with zeros)

$$V_{xx} = -3y \quad V_{yy} = -4x \quad V_{xy} = 3 - 3x - 2y$$

$$d(2/3, 1) = 7 > 0 \quad V_{xx}(2/3, 1) = -3 < 0$$

this is a relative maximum, for dimensions  $(2/3, 1, 2)$



graph of  $z = V$

### Example

A revenue function is:  $R = 500x + 800y + 1.5xy - 1.5x^2 - y^2$ , where  $x$  and  $y$  are prices of two competing products. Find  $x$  and  $y$  to maximize  $R$ .

$$V_x = 500 + 1.5y - 3x \quad V_y = 800 + 1.5x - 2y$$

Solving the system yields one critical point at  $(586.67, 840)$

$$V_{xx} = -3 \quad V_{yy} = -2 \quad V_{xy} = 1.5$$

$$d = 3.75 > 0 \quad V_{xx} < 0 \quad \rightarrow \text{relative maximum}$$

## 13.10 Lagrange multipliers

constraints on optimization problems  $\rightarrow$  optimal solution can occur at a boundary point

e.g., find the rectangle of maximum area inscribed in an ellipse centered at origin:

$$f(x, y) = \text{Area} = (2x)(2y) = 4xy = \text{the objective function}$$

$$\text{constraint equation: fixed level curve of the ellipse } g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

level curves of  $f$  = family of hyperbolas:  $f(x, y) = 4xy = k$

the hyperbolas that intersect the ellipse meet the constraint

to maximize  $f \rightarrow$  hyperbola tangent to ellipse  $\rightarrow$  gradient vectors are parallel

$$\nabla f = \lambda \nabla g \quad \lambda = \text{Lagrange multiplier}$$

### Lagrange's Theorem

$f$  and  $g$  have continuous first partial derivatives

$f$  has extremum at  $(x_0, y_0)$  on smooth constraint curve  $g(x, y) = c$

$$\nabla g(x_0, y_0) \neq 0$$

$$\text{then } \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \quad \lambda \text{ is a real number}$$

### Method of Lagrange Multipliers

1) solve system of equations with *step by step substitution*:

$$f_x(x, y) = \lambda g_x(x, y)$$

$$f_y(x, y) = \lambda g_y(x, y)$$

$$g(x, y) = c$$

2) evaluate  $f$  at each solution point; largest value is a maximum and smallest value is a minimum

Example:  $f = x^2 - y^2$      $x > 0$      $y > 0$     constraint  $g = y - x^2 = 0$

The level curves of  $f$  are  $x^2 - y^2 = k$ , which are hyperbolas symmetric to the  $x$ -axis, with decreasing values of  $k$  as the hyperbolas approach the origin. We want to find the value of  $k$  when the hyperbola just touches the parabola  $y = x^2$ .

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) & 2x &= \lambda(-2x) \rightarrow \lambda = -1 \\ f_y(x, y) &= \lambda g_y(x, y) & -2y &= \lambda \rightarrow -2y = -1 \rightarrow y = 1/2 \\ g &= c & y - x^2 &= 0 \rightarrow 1/2 - x^2 = 0 \rightarrow x = \sqrt{2}/2 \\ f(\sqrt{2}/2, 1/2) &= 1/4 = \max \end{aligned}$$

Example  $f = 100x^{0.6}y^{0.4}$      $g = 48x + 36y = 100,000$     ( $x = \$48$  per unit,  $y = \$36$  per unit)  
Find maximum production  $f$  within cost constraint  $g$ .

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) & 60x^{-4}y^4 &= \lambda(48) \rightarrow \lambda = (15/12)x^{-4}y^4 \\ f_y(x, y) &= \lambda g_y(x, y) & 40x^6y^{-6} &= \lambda(36) \rightarrow \frac{40x^6y^{-6}}{36} = \frac{15x^{-4}y^4}{12} \rightarrow \frac{8x}{9} = y \\ g &= c & 48x + 36y &= 100,000 \rightarrow 48x + 36\left(\frac{8x}{9}\right) = 100,000 \rightarrow x = 1250, y = 10,000/9 \\ f(1250, 10,000/9) &\approx 119,247 \text{ units} \end{aligned}$$

Inside a region example: find extreme values of  $f(x, y) = e^{-xy}$  with constraint  $g = x^2 + y^2 \leq 1$

The level curves of  $f$  are hyperbolas ( $-xy = \ln(c)$ ) in Q1 & Q3 or in Q2 & Q4, depending on  $c$ .

$$\begin{aligned} 1) \text{ on the circle: } & f_x = \lambda g_x & -y e^{-xy} &= \lambda(2x) \\ & f_y = \lambda g_y & -x e^{-xy} &= \lambda(2y) \rightarrow y^2 = x^2 \\ & g = x^2 + y^2 = 1 & x = \pm(1/\sqrt{2}) & y = \pm(1/\sqrt{2}) \\ \text{min's at } & f(1/\sqrt{2}, 1/\sqrt{2}) \text{ and } f(-1/\sqrt{2}, -1/\sqrt{2}) & \approx .607 \\ \text{max's at } & f(1/\sqrt{2}, -1/\sqrt{2}) \text{ and } f(-1/\sqrt{2}, 1/\sqrt{2}) & \approx 1.649 \end{aligned}$$

2) inside the circle – no constraint here, so use partial derivatives

$$\begin{aligned} f_x \text{ and } f_y &= 0 \text{ at } (0, 0) \\ f_{xx} &= y^2 e^{-xy} & f_{yy} &= x^2 e^{-xy} & f_{xy} &= xy e^{-xy} \\ \text{at } (0, 0), & d = 0 \rightarrow \text{saddle point with } f = 1 \end{aligned}$$

**for  $f(x, y, z)$  and  $g(x, y, z)$ , add  $f_z = \lambda g_z$  to the calculations**

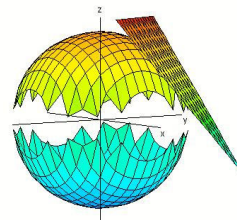
Example: Find the minimum value of  $f(x, y, z) = x^2 + y^2 + z^2$  subject to the constraint  $x + y + z = 9$ .

The level surfaces of  $f$  are spheres centered at the origin. The constraint is a plane. Since  $f$  can increase without bound, the minimum value of  $f$  determines a sphere tangent to the constraint plane.

$$\left. \begin{aligned} f_x &= \lambda g_x & 2x &= \lambda \\ f_y &= \lambda g_y & 2y &= \lambda \\ f_z &= \lambda g_z & 2z &= \lambda \end{aligned} \right\} x = y = z$$

$$x + y + z = 9 = 3x \rightarrow 3 = x = y = z$$

$$f(3, 3, 3) = 27$$



### Lagrange method with 2 constraints

2 constraint functions  $g$  and  $h$ , use 2<sup>nd</sup> Lagrange multiplier  $\mu$  ( $\nabla$ 's not parallel)

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Example: Find min temperature  $T(x, y, z) = x^2 + y^2 + z^2$  on surface  $g: x + 2z = 4$  on line of intersection with surface  $h: x + y = 8$ .

$$\nabla T = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \quad \lambda \nabla g = \lambda\mathbf{i} + 2\lambda\mathbf{k} \quad \mu \nabla h = \mu\mathbf{i} + \mu\mathbf{j}$$

$$\begin{aligned} 2x &= \lambda + \mu \rightarrow 2x - \mu = \lambda \\ 2y &= \mu \rightarrow 2x - 2y = \lambda \\ 2z &= 2\lambda \rightarrow z = \lambda = 2x - 2y \end{aligned}$$

Use the relationship  $z = 2x - 2y$  to substitute into the system of constraints:

$$\left. \begin{aligned} x + 2z &= 4 \\ x + y &= 8 \end{aligned} \right\} \left. \begin{aligned} x + 2(2x - 2y) &= 4 \\ x + y &= 8 \end{aligned} \right\} \begin{aligned} 5x - 4y &= 4 \\ x + y &= 8 \end{aligned}$$

$$\rightarrow x = 4, y = 4, z = 0 \quad f(4, 4, 0) = 32$$

Example: Maximize  $f(x, y, z) = xyz$  subject to constraints  $g(x, y, z): x^2 + z^2 = 5$  and  $h(x, y, z): x - 2y = 0$ .

$$\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \quad \nabla g = 2x\mathbf{i} + 2z\mathbf{k} \quad \nabla h = \mathbf{i} - 2\mathbf{k}$$

$$\begin{aligned} yz &= 2\lambda x + \mu \\ xz &= -2\mu \rightarrow \mu = \frac{-xz}{2} \\ xy &= 2\lambda z \rightarrow \lambda = \frac{xy}{2z} \end{aligned} \quad \left| \begin{array}{l} \text{constraints} \\ z = \sqrt{5 - x^2} \\ y = \frac{x}{2} \end{array} \right.$$

$$yz = 2x \left( \frac{xy}{2z} \right) - \frac{xz}{2}$$

$$y\sqrt{5 - x^2} = \frac{x^2 y}{\sqrt{5 - x^2}} - \frac{x\sqrt{5 - x^2}}{2} \rightarrow \frac{x}{2} \sqrt{5 - x^2} = \frac{x^3}{2\sqrt{5 - x^2}} - \frac{x\sqrt{5 - x^2}}{2}$$

$$\rightarrow 0 = x(3x^2 - 10) \rightarrow x = 0, \sqrt{\frac{10}{3}}$$

$$\begin{array}{ll}
 x = 0 & x = \sqrt{\frac{10}{3}} \\
 y = 0 & y = \frac{1}{2}\sqrt{\frac{10}{3}} \\
 z = \sqrt{5} & z = \sqrt{\frac{5}{3}} \\
 xyz = 0 & xyz = \frac{5\sqrt{15}}{9} = \max
 \end{array}$$

Example: Find the highest point on the curve of intersection between the sphere  $x^2 + y^2 + z^2 = 36$  and the plane  $2x + y - z = 2$ .

Maximize  $f(x, y, z) = z$

Constraints:  $g(x, y, z): x^2 + y^2 + z^2 = 36$

$h(x, y, z): 2x + y - z = 2$

$$\left. \begin{array}{l}
 f_x = 0 = 2x\lambda + 2\mu \\
 f_y = 0 = 2y\lambda + \mu \\
 f_z = 1 = 2z\lambda - \mu
 \end{array} \right\} x = 2y$$

h:  $z = 2x + y - 2 \rightarrow z = 5y - 2$

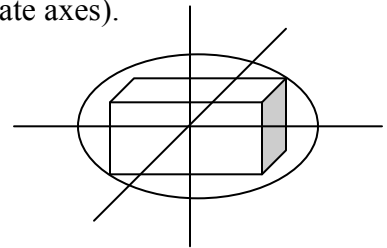
g:  $(2y)^2 + y^2 + (5y - 2)^2 = 36 \rightarrow y = \frac{5 + \sqrt{265}}{15}$  (choose the positive root)

The point of maximum  $z$  is  $\left( \frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3} \right)$

Example: Find the dimensions of a rectangular box of maximum volume that can be inscribed

in the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  (edges parallel to the coordinate axes).

Maximize volume  $V = 2x2y2z = 8xyz$



$$\left. \begin{array}{l}
 f_x: 8yz = 2x\lambda/a^2 \quad \lambda = \frac{4a^2 yz}{x} \\
 f_y: 8xz = 2y\lambda/b^2 \quad \lambda = \frac{4b^2 xz}{y} \\
 f_z: 8xy = 2z\lambda/c^2 \quad \lambda = \frac{4c^2 xy}{z}
 \end{array} \right\} x^2 = \frac{a^2}{b^2} y^2, \quad z^2 = \frac{c^2}{b^2} y^2$$

$\frac{a^2 y^2}{b^2 a^2} + \frac{y^2}{b^2} + \frac{c^2 y^2}{b^2 c^2} = 1 \rightarrow y = \frac{\sqrt{3} b}{3}$

The dimensions are  $\left( \frac{2\sqrt{3} a}{3}, \frac{2\sqrt{3} b}{3}, \frac{2\sqrt{3} c}{3} \right)$  and the volume is  $\frac{8\sqrt{3}}{9}(abc)$ .