NOTES – Vector-Valued Functions (Chpt 12)

12.1 Space curves and vector-valued functions

vector-valued functions: maps real numbers onto vectors; component functions real-valued

plane curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ or $\mathbf{r}(t) = \langle f(t), g(t) \rangle$

space curve $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

curve = set of points <u>and</u> defining equations

$\mathbf{r} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ $\neq$ $\mathbf{r} = \sin(t^2)\,\mathbf{i} + \cos(t^2)\,\mathbf{j}$

unit circle, period 2π unit circle, period $\sqrt{2\pi}$

domain of **r** is intersection of domains of f, g, and h

<u>Example</u>: $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 - t)\,\mathbf{j}$ $0 \le t \le \infty$

$t = x^2$ $y = 1 - x^2$ $\rightarrow$ graphs as a parabola, $x \ge 0$

<u>Example</u>: $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\,\mathbf{k}$

$x^2 + y^2 = 16$ $\rightarrow$ curve on right circular cylinder, radius 4, centered on z axis

$z = t$ $\rightarrow$ helix

<u>Example</u>: convert $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ to a vector-valued function

$\dfrac{x}{5} = \cos t$ $\dfrac{y}{4} = \sin t$ $\mathbf{r}(t) = 5\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}$ (<u>or</u> $\mathbf{r}(t) = 5\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j}$)

<u>Example</u>: intersection of semiellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{4} = 1$ and parabolic cylinder $y = x^2$

solve for z: $z = \dfrac{1}{2}\sqrt{16 - 4x^2 - y^2}$ parametrics: $x = t$, $y = t^2$

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{2}\sqrt{16 - 4t^2 - t^4}\,\mathbf{k}$$

Limits and continuity

$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k}$ if f, g, h have limits as $t \rightarrow a$

vector versions of limit theorems; orientation of curve $\mathbf{r}(t)$ used to define on-sided limits

$\mathbf{r}(t)$ is continuous at $t = a$ if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

- continuous on interval I if continuous at every point in I

<u>example</u>: $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \ln t\,\mathbf{k}$ continuous for $t > 0$

12.2 Differentiation of vector-valued functions $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$

the curve is smooth on an open interval I if f ', g', and h' are continuous on I and $\mathbf{r}'(t) \neq 0$ in I.

Example: $\mathbf{r}(t) = \dfrac{3t}{1+t^3}\mathbf{i} + \dfrac{3t^2}{1+t^3}\mathbf{j}$ $\qquad \mathbf{r}'(t) = \dfrac{3-6t^3}{(1+t^3)^2}\mathbf{i} + \dfrac{6t-3t^4}{(1+t^3)^2}\mathbf{j}$

there is no value of t that makes $\mathbf{r}'(t) = 0$; there is a discontinuity at $t = -1$
the curve is smooth on $(-\infty, -1)$ and $(-1, \infty)$

$D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$	$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
$D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$	$D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
$D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$	if $\mathbf{r}(t) \bullet \mathbf{r}(t) = c$, then $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$
$D_t[\mathbf{r}(t) \bullet \mathbf{u}(t)] = \mathbf{r}(t) \bullet \mathbf{u}'(t) + \mathbf{r}'(t) \bullet \mathbf{u}(t)$	

Integration of vector-valued functions

plane curve: $\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j}$

space curve: $\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}$

indefinite integral produces a family of vector-valued functions

$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$ $\qquad$ where $\mathbf{C} = C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}$

12.2 Velocity and acceleration

position vector $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$
$\mathbf{r}'(t)$ = velocity vector or tangent vector; indicates direction of motion
$\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ = speed
$\mathbf{r}''(t)$ = acceleration

if motion is at a constant speed, then $\mathbf{r}'(t) \perp \mathbf{r}''(t)$

example: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ $\qquad \left[\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1\right]$

velocity $\quad \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$

speed $\quad \|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t}$

acceleration $\quad \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

Example: $\mathbf{a}(t) = \mathbf{i} + \mathbf{k}$ $\mathbf{v}(0) = 5\mathbf{j}$ $\mathbf{r}(0) = 0$ find $\mathbf{r}(2)$

$$\mathbf{v}(t) = \int \mathbf{a}(t) = (t + C_1)\,\mathbf{i} + C_2\,\mathbf{j} + (t + C_3)\,\mathbf{k}$$

$$\mathbf{v}(0) = 5\mathbf{j} \rightarrow C_1 = 0, C_3 = 0, C_2 = 5 \rightarrow \mathbf{v}(t) = t\,\mathbf{i} + 5\,\mathbf{j} + t\,\mathbf{k}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t) = \left(\frac{t^2}{2} + C_4\right)\mathbf{i} + (5t + C_5)\,\mathbf{j} + \left(\frac{t^2}{2} + C_6\right)\mathbf{k}$$

$$\mathbf{r}(0) = 0 \rightarrow C_4 = 0, C_5 = 0, C_6 = 0 \rightarrow \mathbf{r}(t) = \left(\frac{t^2}{2}\right)\mathbf{i} + (5t)\,\mathbf{j} + \left(\frac{t^2}{2}\right)\mathbf{k}$$

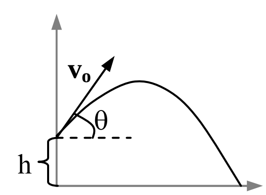
$$\mathbf{r}(2) = 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}$$

Projectile motion

gravity only force acting on a projectile after launch

$\mathbf{F} = -mg\,\mathbf{j}$ $\mathbf{a} = -g\,\mathbf{j}$ ($g = 32$ ft/s^2 or 9.81 m/s^2)

position vector: $\mathbf{r}(t) = -\frac{1}{2} gt^2\,\mathbf{j} + t\mathbf{v_o} + \mathbf{r_o}$



$\|\mathbf{v_o}\| = v_o$ $\mathbf{v_o} = v_o\cos\theta\,\mathbf{i} + v_o\sin\theta\,\mathbf{j}$

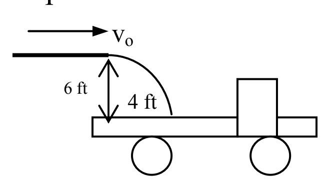
$\|\mathbf{r_o}\| = h$ $\mathbf{r_o} = h\,\mathbf{j}$

given h, v_o, θ: $\mathbf{r}(t) = (v_o\cos\theta)t\,\mathbf{i} + [h + (v_o\sin\theta)t - \frac{1}{2} gt^2]\,\mathbf{j}$

for a projectile fired from ground level, the range $x = \dfrac{v_0^2 \sin 2\theta}{g} = \dfrac{t^2 g \cot\theta}{2}$

rectangular form of position function: $y = -\dfrac{16\sec^2\theta}{v_0^{\,2}}x^2 + (\tan\theta)x + h$

Example Gravel is coming off a conveyor belt and falling 6 ft into the center of a pickup truck bed, which is 4 ft away. At what speed must it come off the belt to hit this spot?



initial angle $\theta = 0^o$ $\cos\theta = 1$ $\sin\theta = 0$ $h = 6$ ft

$\mathbf{r}(t) = v_o t\mathbf{i} + (6 - 16t^2)\mathbf{j}$

when x = 4, y = 0 $\rightarrow$ solve for *t* when $6 - 16t^2 = 0 \rightarrow t = \sqrt{3/8}$

solve for v_o where $v_o t = 4 \rightarrow v_o\sqrt{3/8} = 4 \rightarrow v_o = \dfrac{8\sqrt{6}}{3} \approx 6.53$ ft/sec

Using the rectangular form (with y = 0): $0 = -\dfrac{16}{v_0^{\,2}}x^2 + 6 \rightarrow v_o = \sqrt{16^2/6} \approx 6.53$

12.4 Tangent vectors and normal vectors

If curve C is smooth on open interval I (**r'** is continuous and nonzero), then the

unit tangent vector at t is: $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

The tangent line at a point on the curve is a line passing through the point and parallel to **T**(t)

<u>Example</u> Find **T**(t) and the parametric equations for a tangent line at the point for t = 1

$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2/3\,\mathbf{k}$ $\qquad$ $t = 1 \rightarrow$ point (1, 1, 2/3)

$\mathbf{T}(t) = \dfrac{\mathbf{i}+2t\mathbf{j}}{\|\mathbf{r}'(t)\|}$ $\qquad$ at t = 1, $\mathbf{T}(t) = \dfrac{1}{\sqrt{5}}(\mathbf{i}+2\mathbf{j})$ $\qquad$ direction numbers = $\langle 1,2,0\rangle$

($\langle a,b,c\rangle$)

parametric equations: $\quad x = 1 + s \qquad y = 1 + 2s \qquad z = 2/3$

principal unit normal vector

if $\mathbf{T}'(t) \neq 0$, the principal unit normal vector at t is: $\quad \mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

<u>Example</u> Find **N**(t) and **N**(1) for $\mathbf{r}(t) = 4t^2\,\mathbf{i} + 2t\,\mathbf{j}$

$\mathbf{r}'(t) = 8t\,\mathbf{i} + 2\,\mathbf{j} \qquad \|\mathbf{r}'\| = \sqrt{64t^2+4} \qquad \mathbf{T}(t) = \dfrac{8t\,\mathbf{i}+2\,\mathbf{j}}{\sqrt{64t^2+4}} = \dfrac{1}{\sqrt{16t^2+1}}(4t\,\mathbf{i}+\mathbf{j})$

$D_t[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t) + f'(t)\mathbf{u}(t)$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{16t^2+1}}(4\,\mathbf{i}) - \frac{16t}{(16t^2+1)^{3/2}}(4t\,\mathbf{i}+\mathbf{j}) = \left[\frac{4}{\sqrt{16t^2+1}} - \frac{64t^2}{(16t^2+1)^{3/2}}\right]\mathbf{i} - \frac{16t}{(16t^2+1)^{3/2}}\mathbf{j}$$

$$= \frac{4}{(16t^2+1)^{3/2}}\mathbf{i} - \frac{16t}{(16t^2+1)^{3/2}}\mathbf{j}$$

$$\|\mathbf{T}'\| = \sqrt{\frac{16}{(16t^2+1)^3} + \frac{256t^2}{(16t^2+1)^3}} = \frac{4}{16t^2+1}j$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{16t^2+1}{4}\left[\frac{4}{(16t^2+1)^{3/2}}\mathbf{i} - \frac{16t}{(16t^2+1)^{3/2}}\,\mathbf{j}\right] = \frac{1}{\sqrt{16t^2+1}}(\mathbf{i}-4t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{17}}(\mathbf{i}-4t\mathbf{j})$$

A unit normal vector is orthogonal to **T**; the <u>principal</u> **N** points toward the concave side of a plane curve or in the direction a space curve is turning. **T** points in the direction the object is moving.

For plane curves: $\mathbf{T}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$

unit normal vectors are: $\mathbf{N_1}(t) = y(t)\,\mathbf{i} - x(t)\,\mathbf{j}$ or $\mathbf{N_2}(t) = -\,y(t)\,\mathbf{i} + x(t)\,\mathbf{j}$

Tangential and normal components of acceleration

if speed is constant, $\mathbf{v} \perp \mathbf{a}$ ($\mathbf{a}$ is not in direction of motion)
if speed variable, $\mathbf{a}$ has components and is in the plane determined by $\mathbf{T}$ and $\mathbf{N}$

$$\mathbf{a}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$$

if $\mathbf{r}(t)$ is the position vector for a smooth curve C,

$$a_T = D_t[\|\mathbf{v}\|] = \mathbf{a}\bullet\mathbf{T} = \frac{\mathbf{v}\bullet\mathbf{a}}{\|\mathbf{v}\|} \qquad a_N = \|\mathbf{v}\|\,\|\mathbf{T}'\| = \mathbf{a}\bullet\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_T{}^2}$$

$a_N \geq 0$; also called centripetal component of acceleration

<u>Example</u> Find the acceleration components for $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$ $\quad \mathbf{a}(t) = 2\,\mathbf{j}$ $\quad \|\mathbf{a}\| = 2$

$$a_T = \frac{\mathbf{v}\bullet\mathbf{a}}{\|\mathbf{v}\|} = \frac{\langle 1,2t\rangle\bullet\langle 0,2\rangle}{\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$$

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T{}^2} = \sqrt{4 - \frac{16t^2}{1+4t^2}} = \frac{2}{\sqrt{1+4t^2}}$$

12.5 Arc length

If C is a smooth curve on [a, b] and $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$

$$\text{arc length } s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt$$

arc length parameter

arc length function: $s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$

s = arc length parameter
$\mathbf{r}$ can be expressed as a function of s

<u>Example</u>

$\mathbf{r}(t) = (4 - 4t)\,\mathbf{i} + 2t\,\mathbf{j}$ $\quad \mathbf{r}'(t) = -4\,\mathbf{i} + 2\,\mathbf{j}$ $\quad \|\mathbf{r}'\| = 2\sqrt{5}$ $\quad 0 \leq t \leq 1$

$$s(t) = \int_0^t 2\sqrt{5}\,du = 2\sqrt{5}\,t \qquad t = \frac{s}{2\sqrt{5}}$$

$$\mathbf{r}(t) = \left(4 - \frac{2s}{\sqrt{5}}\right)\mathbf{i} + \frac{s}{\sqrt{5}}\,\mathbf{j} \qquad 0 \le s \le 2\sqrt{5}$$

$\|\mathbf{r}'(s)\| = 1$ arc length from a to b $= \int_a^b \|\mathbf{r}'(s)\|\,ds = \int_a^b ds = b - a$

if t is any parameter and $\|\mathbf{r}'(s)\| = 1$, then t is the arc length parameter

curvature

arc length parameter $\rightarrow$ curvature, rate of change of **T** with respect to s

curvature $K = \left\|\frac{d\mathbf{T}}{ds}\right\| = \|\mathbf{T}'(s)\|$ (for a circle $K = 1/r$)

if C is a smooth curve, the curvature at t: $K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

curvature in rectangular coordinates

if C is a plane curve $y = f(x)$ and is twice-differentiable, the curvature at (x, y) is:

$$K = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$$

if C is a curve with curvature K at point P, the circle passing through P with $r = 1/K$ is the circle of curvature and lies on the concave side of C. r = radius of curvature at P; circle center = center of curvature.

can graphically estimate K from the radius of a circle passing through P: $K = 1/r$

acceleration, speed, and curvature

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N} \qquad \frac{ds}{dt} = \text{speed} = \|\mathbf{r}'(t)\|$$

Tangential component of acceleration is the rate of change of the speed, which is the rate of change of arc length. Normal component is a function of both speed and curvature.

Example Find a_T and a_N for: $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$

$\mathbf{r}'(t) = 2t\,\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}$ $\quad \|\mathbf{r}'(t)\| = 2\sqrt{2t^2+1} = ds/dt$ $\quad a_T = \frac{d^2s}{dt^2} = \frac{4t}{\sqrt{2t^2+1}}$

$K = \frac{\sqrt{2}}{2(2t^2+1)^{3/2}}$ $\quad K(ds/dt)^2 = \frac{2\sqrt{2}}{\sqrt{2t^2+1}} = a_N$

frictional force

for an object moving in contact with a stationary object:

$$\mathbf{F} = m\mathbf{a} = m\left(\frac{d^2s}{dt^2}\right)\mathbf{T} + mK\left(\frac{ds}{dt}\right)^2\mathbf{N} \qquad \text{friction is part of total force}$$

for a car rounding a turn at constant speed, friction keeps it from sliding off the road

- **magnitude = $\mathbf{a_N}$**

<u>Example</u> Find the frictional force keeping on the road a 4000 lb car traveling at 30 mph around a circular interchange with r = 100 ft

$K = 1/r = 1/100 \qquad ds/dt = 44$ ft/s

$$F_N = mK\left(\frac{ds}{dt}\right)^2 = \frac{(4000\,\text{lb})(44\,\text{ft}/\text{s})^2}{100\,\text{ft}} = 77{,}440 \text{ ft-lb/s}^2$$