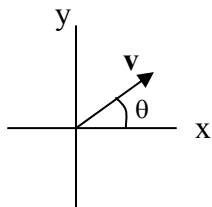


## NOTES – Vectors and the Geometry of Space (Chapt 11)

### 11.1 Vectors – 2D

$$\mathbf{u} = \langle u_1, u_2 \rangle \quad \mathbf{v} = \langle v_1, v_2 \rangle \quad \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$



$$\mathbf{v} = \|\mathbf{v}\| \cos\theta \mathbf{i} + \|\mathbf{v}\| \sin\theta \mathbf{j}$$

### 11.2 Vectors – 3D

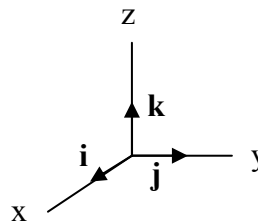
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

if  $\mathbf{u} = c\mathbf{v}$  then the vectors are parallel

if point  $P = (x_1, y_1, z_1)$  and  $Q = (x_2, y_2, z_2)$

then the vector  $\overline{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$P, Q,$  and  $R$  are points on the same line if  $\overline{PQ} \parallel \overline{PR}$  that is,  $\overline{PQ} = c\overline{PR}$



application: tripod supporting a 120 lb camera; forces are equal on the legs

$$\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$$

if  $\mathbf{F}_1 = c\langle 0, -1, -4 \rangle$   $\mathbf{F}_2 = c\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \rangle$   $\mathbf{F}_3 = c\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \rangle$  and total force  $\mathbf{F} = -120\mathbf{k}$ ,

then  $c = 10$  and  $\mathbf{F}_1 = \langle 0, -10, -40 \rangle$   $\mathbf{F}_2 = \langle 5\sqrt{3}, 5, -40 \rangle$   $\mathbf{F}_3 = \langle -5\sqrt{3}, 5, -40 \rangle$

### 11.3 Dot product

$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$  (inner or scalar product)  $[\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}; \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2]$

if  $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ , then  $\mathbf{u} \cdot \mathbf{v} = \cos\theta$  (the angle between them)

if  $\|\mathbf{u}\| \neq \|\mathbf{v}\|$ , the  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$  or  $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$

if  $\mathbf{u} \perp \mathbf{v}$ , then  $\mathbf{u} \cdot \mathbf{v} = 0$

#### Direction cosines

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

for angles between  $\mathbf{v}$  and unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ :

bet.  $\mathbf{v}$  and  $\mathbf{i}$

bet.  $\mathbf{v}$  and  $\mathbf{j}$

bet.  $\mathbf{v}$  and  $\mathbf{k}$

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|}$$

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

normalized vector:  $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|} \mathbf{i} + \frac{v_2}{\|\mathbf{v}\|} \mathbf{j} + \frac{v_3}{\|\mathbf{v}\|} \mathbf{k} = \cos\alpha \mathbf{i} + \cos\beta \mathbf{j} + \cos\gamma \mathbf{k}$

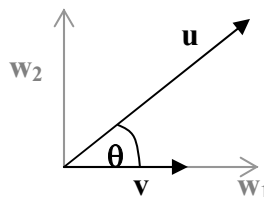
this is a unit vector, so  $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

## Projections and vector components

the projection of  $\mathbf{u}$  onto  $\mathbf{v}$ , or the vector component

of  $\mathbf{u}$  along  $\mathbf{v}$ :  $\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u}$

the vector component of  $\mathbf{u} \perp \mathbf{v}$ :  $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$



$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (k) \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$k = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\| \cos \theta = (\text{scalar}) \text{ component of } \mathbf{u} \text{ in the direction of } \mathbf{v}$$

### Example

$$\mathbf{u} = \langle 3, -5, 2 \rangle \quad \mathbf{v} = \langle 7, 1, -2 \rangle$$

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{21 - 5 - 4}{54} \right) \mathbf{v} = \frac{2}{9} \langle 7, 1, -2 \rangle = \left\langle \frac{14}{9}, \frac{2}{9}, -\frac{4}{9} \right\rangle$$

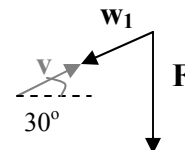
### Example

A 600 lb boat sits on a  $30^\circ$  incline. How much force is required to keep the boat on the ramp?

$$\mathbf{F} = -600 \mathbf{j} \quad \text{let } \mathbf{v} \text{ be a unit vector along the ramp: } \mathbf{v} = \cos 30 \mathbf{i} + \sin 30 \mathbf{j} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

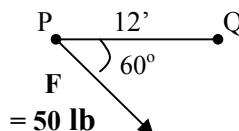
$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{F} = \left( \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = (\mathbf{F} \cdot \mathbf{v}) \mathbf{v} = (-600)(1/2) \mathbf{v} = -300 \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

-300 = scalar component of  $\mathbf{F}$  in direction of  $\mathbf{v} \rightarrow$  300 lbs is required to keep the boat on the ramp



**Work = (force)(distance)**

$$W = \|\text{proj}_{\overline{PQ}} \mathbf{F}\| \|\overline{PQ}\| = \mathbf{F} \cdot \overline{PQ}$$



$\mathbf{F}$  pulls weight from P to Q

$$\mathbf{F} \cdot \overline{PQ} = \langle 25, -25\sqrt{3} \rangle \cdot \langle 12, 0 \rangle = 300 \text{ ft-lb}$$

## 11.4 Cross product (vector product) $\rightarrow$ vector orthogonal to two other vectors

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$$

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

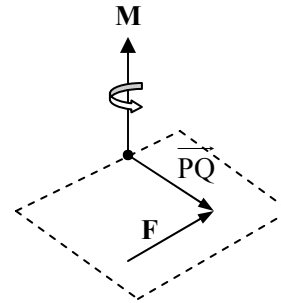
$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$	$\mathbf{u} \times \mathbf{u} = \mathbf{0}$
$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$	$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
$c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$	$\ \mathbf{u} \times \mathbf{v}\  = \ \mathbf{u}\  \ \mathbf{v}\  \sin \theta$
$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$	if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ , $\mathbf{u}$ & $\mathbf{v}$ are multiples

$\|\mathbf{u} \times \mathbf{v}\| = \text{area of parallelogram with } \mathbf{u}, \mathbf{v} \text{ as sides}$

Torque (moment  $\mathbf{M}$  of force  $\mathbf{F}$ ):  $\mathbf{M} = \overline{\mathbf{PQ}} \times \mathbf{F}$



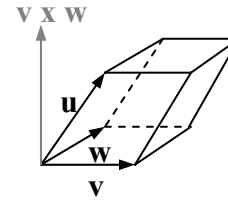
### Triple scalar product

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \text{volume of a parallelepiped with } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \text{ as adjacent sides}$

$\|\mathbf{v} \times \mathbf{w}\| = \text{area of base}$

$\|\text{proj}_{\mathbf{v} \times \mathbf{w}} \mathbf{u}\| = \text{height}$



$V = 0$  if all 3 vectors are coplanar (have same initial point, lie in same plane)

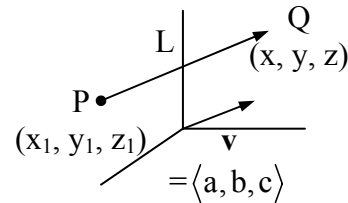
### 11.5 Lines & planes in space

a line  $L$  ( $\overline{\mathbf{PQ}}$ ) is parallel to a direction vector  $\mathbf{v} = \langle a, b, c \rangle$

$a, b, c$  are direction numbers

$\overline{\mathbf{PQ}} = t\mathbf{v}$  (scalar multiple)

$\overline{\mathbf{PQ}} = \langle at, bt, ct \rangle$



### Parametric equations of a line in space

line  $L$  through  $P(x_1, y_1, z_1)$  parallel to  $\mathbf{v} = \langle a, b, c \rangle$

$$x = x_1 + at \quad y = y_1 + bt \quad z = z_1 + ct$$

**Symmetric equations of line:**  $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

parametric and symmetric equations are not unique: different  $t$  values  $\rightarrow$  different point  
 $\rightarrow$  different parametric equations

### Equation of a plane in space

obtain from a point in the plane and a normal vector

point  $P(x_1, y_1, z_1)$  in plane    any other point in plane  $Q(x, y, z)$     normal vector  $\mathbf{n} = \langle a, b, c \rangle$

$$\mathbf{n} \cdot \overline{\mathbf{PQ}} = \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

**standard equation:**  $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

**general equation:**  $ax + by + cz + d = 0$     from general form,  $\mathbf{n} = \langle a, b, c \rangle$

From 3 points in a plane: obtain 2 vectors in the plane,  $\mathbf{u}$  and  $\mathbf{v}$ .  $\mathbf{u} \times \mathbf{v} = \mathbf{n} = \langle a, b, c \rangle$   
 use any point to get standard and general equations

Example Find the equation of the plane that goes through the points (3, -1, 1) and (2, 3, 1) and is perpendicular to the plane  $2x - 3y - z = 5$ .

A vector  $\mathbf{u}$  connecting the two points is  $\langle 1, -4, 0 \rangle$ . The vector  $\mathbf{n}_1$  normal to the second plane is  $\langle 2, -3, -1 \rangle$ . Since the planes are perpendicular,  $\mathbf{n}_1$  is parallel to the first plane. To find a vector normal to the first plane, do the cross product between the vector  $\mathbf{u}$  in the plane and  $\mathbf{n}_1$ .

$$\mathbf{u} \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} + 5\mathbf{k} \quad \rightarrow \quad \mathbf{n}_2 = \langle 4, 1, 5 \rangle = \langle a, b, c \rangle$$

pick one point:  $4(x - 3) + (y + 1) + 5(z - 1) = 0$   
 $\rightarrow 4x + y + 5z = 16$  is the equation of the first plane.

Planes are either parallel or intersect at a line; the angle between them is the angle between their normal vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$ :  $\mathbf{n}_1 \perp \mathbf{n}_2$  if  $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$   $\mathbf{n}_1 \parallel \mathbf{n}_2$  if  $\mathbf{n}_1 = k\mathbf{n}_2$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (0 \leq \theta \leq \pi/2)$$

Example

Find angle of intersection of two planes and parametric equations of the line of intersection

planes:  $x - 3y + 6z = 0$   $\mathbf{n}_1 = \langle 1, -3, 6 \rangle$ ;  $5x + y - z = 0$   $\mathbf{n}_2 = \langle 5, 1, -1 \rangle$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{4}{\sqrt{46}\sqrt{27}} \quad \theta \approx 83.5^\circ$$

To find line of intersection, solve the equations of the planes

$$x - 3y + 6z = 0$$

$$5x + y - z = 0$$

use result to solve for y:

$$y = \frac{31z}{16}$$

$$16x + 3z = 0 \rightarrow x = -\frac{3z}{16}$$

let  $t = -\frac{3z}{16} \rightarrow x = t$   $y = -\frac{31}{3}t$   $z = -\frac{16}{3}t$  direction numbers =  $\left\langle 1, -\frac{31}{3}, -\frac{16}{3} \right\rangle$

## Sketching planes

Trace: line of intersection of plane with coordinate planes, i.e., xy plane, xz plane, yz plane.

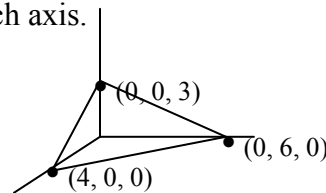
$$3x + 2y + 4z = 12: \quad z = 0 \rightarrow 3x + 2y = 12; \quad y = 0 \rightarrow 3x + 4z = 12; \quad x = 0 \rightarrow 2y + 4z = 12$$

Easy way: set 2 variables equal to zero and solve for point of intersection on each axis.

$$\rightarrow \text{x-axis: } (4, 0, 0); \quad \text{y-axis: } (0, 6, 0); \quad \text{z-axis: } (0, 0, 3)$$

If one variable is missing in equation, plane is parallel to the axis of the missing variable.

If two variables are missing, plane is parallel to the plane of the missing two variables.



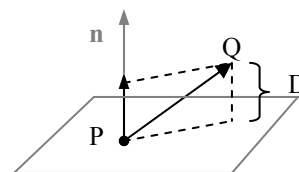
## Distances

### Between point Q and a plane

find a point P in the plane  $ax + by + cz + d = 0$  by letting  $y = 0$

$$\text{and } z = 0 \rightarrow ax + d = 0 \quad P = (-d/a, 0, 0)$$

$$\text{find } \overrightarrow{PQ} \text{ and } \mathbf{n}; \quad \text{distance } D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$



### Example

$$Q = (1, 2, 3) \quad \text{plane: } 2x - y + z = 4$$

$$\mathbf{n} = \langle 2, -1, 1 \rangle$$

$$y = 0, z = 0 \rightarrow x = 2 \rightarrow P = (2, 0, 0)$$

$$\overrightarrow{PQ} = \langle -1, 2, 3 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

Distance between a point  $Q(x_0, y_0, z_0)$  and a plane  $ax + by + cz + d = 0$  can also be found by:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

### Between parallel planes

Find a point in one plane; use a, b, c, and d from the second plane

$$\text{Example} \quad \text{plane1: } 3x - y + 2z - 6 = 0$$

$$\text{plane2: } 6x - 2y + 4z + 4 = 0$$

$$P \text{ in plane1: } y = 0, z = 0 \rightarrow P(2, 0, 0)$$

$$\text{plane2: } a = 6, b = -2, c = 4, d = 4$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|6(2) + 4|}{\sqrt{36 + 4 + 16}} = \frac{8}{\sqrt{56}} = \frac{8}{2\sqrt{14}} = \frac{4}{\sqrt{14}}$$

### Between a point and a line

For a point P on a line  $\mathbf{u}$ , and a point Q, distance D between Q and the line:

$$D = \frac{\|\overrightarrow{\mathbf{PQ}} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \|\overrightarrow{\mathbf{PQ}}\| \sin \theta$$

Example Find the distance between Q(3, -1, 4) and the line given by the following parametric equations:  $x = -2 + 3t$   $y = -2t$   $z = 1 + 4t$   $\mathbf{u} = \langle 3, -2, 4 \rangle$  (direction numbers)

to find point P, let  $t = 0 \rightarrow P(-2, 0, 1)$   $\overrightarrow{\mathbf{PQ}} = \langle 5, -1, 3 \rangle$

$$\overrightarrow{\mathbf{PQ}} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = \langle 2, -11, -7 \rangle \quad \frac{\|\overrightarrow{\mathbf{PQ}} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{174}}{\sqrt{29}} = \sqrt{6}$$

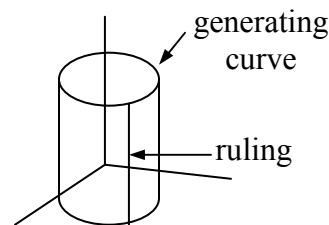
### 11.6 Surfaces in space

sphere:  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

plane:  $ax + by + cz + d = 0$

#### Cylindrical surfaces

If you move a vertical line around the circle  $x^2 + y^2 = a^2$ , it generates a right circular cylinder; the circle is the generating curve (or directrix); the line is called a ruling.



Cylindrical surfaces do not have to be circular!

If C is a curve in a plane and L is a line not in a parallel plane, then the set of all lines parallel to L (rulings) and intersecting C (the generating curve) is called a cylinder.

For right cylinders, the rulings are perpendicular to the coordinate plane containing C and parallel to the axis not included in the equation of the generating curve.

- the equation of the cylinder is the equation of the generating curve.

for  $z = y^2$ , the cylinder is a parabola in the yz plane and the rulings are parallel to the x- axis

for  $z = \sin x$ , the cylinder is a sine curve in the xz plane and the rulings are parallel to the y-axis

#### Surfaces of revolution

If a generating curve is rotated around an axis, it forms a surface of revolution.

##### Equations of surfaces of revolution

around x-axis  $y^2 + z^2 = [r(x)]^2$   
 around y-axis  $x^2 + z^2 = [r(y)]^2$   
 around z-axis  $x^2 + y^2 = [r(z)]^2$

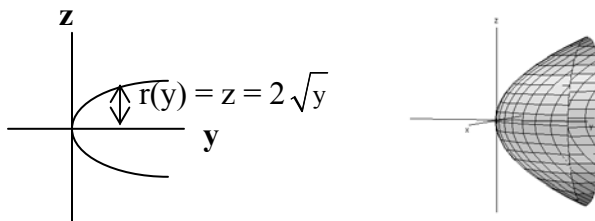
##### Generating curves

either  $y = r(x)$  or  $z = r(x)$   
 either  $x = r(y)$  or  $z = r(y)$   
 either  $x = r(z)$  or  $y = r(z)$

Example Find the equation of the surface formed by revolving  $z^2 = 4y$  about the y axis.

Do a sketch in y-z coordinate system to find r:

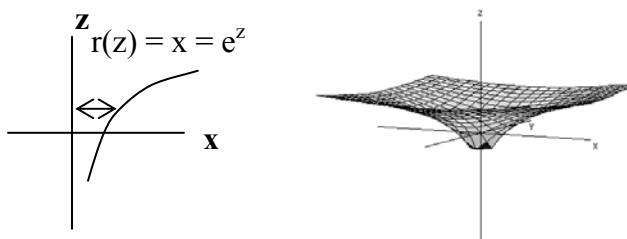
$$x^2 + z^2 = [2\sqrt{y}]^2 \rightarrow x^2 + z^2 = 4y$$



Example Find the equation of the surface formed by rotating  $z = \ln x$  about the z axis.

Do a sketch in x-z coordinate system to find r:

$$x^2 + y^2 = [e^z]^2 \rightarrow x^2 + y^2 = e^{2z}$$



To find a generating curve, backtrack:

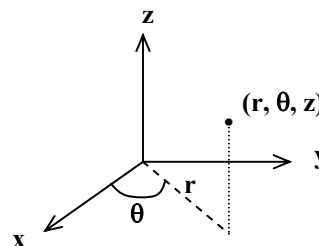
for surface  $x^2 + 3y^2 + z^2 = 9$ , rearrange to get coefficients of 1 on the left side:

$$x^2 + z^2 = 9 - 3y^2 \rightarrow x = r(y) = \sqrt{9 - 3y^2} \text{ or } z = r(y) = \sqrt{9 - 3y^2}$$

- these are both generating curves

### 11.7 Cylindrical coordinates $(r, \theta, z)$

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta & z &= z \\ r^2 &= x^2 + y^2 & \tan \theta &= y/x & z &= z \end{aligned}$$



**for surfaces:** cylinder:  $r = 3$  ( $x^2 + y^2 = 9$ )  
cone:  $r = z$  ( $x^2 + y^2 = z^2$ )

paraboloid:  $r = 2\sqrt{z}$  ( $x^2 + y^2 = 4z$ )  
hyperboloid:  $r^2 = z^2 + 1$  ( $x^2 + y^2 - z^2 = 1$ )

**vertical half-plane:**  $\theta = c$

**horizontal plane:**  $z = c$

#### conversions

parabolic cone  $y^2 = x$

$$r^2 \sin^2 \theta = r \cos \theta \rightarrow r(r \sin^2 \theta - \cos \theta) = 0 \rightarrow r \sin^2 \theta - \cos \theta = 0$$

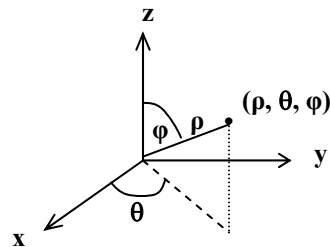
$$r = \frac{\cos \theta}{\sin^2 \theta} = \csc \theta \cot \theta$$

$$\begin{aligned} \text{hyperboloid of 2 sheets } r^2 \cos 2\theta + z^2 + 1 = 0 &\rightarrow r^2(\cos^2 \theta - \sin^2 \theta) + z^2 + 1 = 0 \\ \rightarrow x^2 - y^2 + z^2 = -1 &\rightarrow y^2 - x^2 - z^2 = 1 \end{aligned}$$

**Spherical coordinates**  $(\rho, \theta, \varphi)$

spherical  $\rightarrow$  rectangular  $x = \rho \sin \varphi \cos \theta$   
 $y = \rho \sin \varphi \sin \theta$   
 $z = \rho \cos \varphi$

rectangular  $\rightarrow$  spherical  $\rho^2 = x^2 + y^2 + z^2$   
 $\tan \theta = y/x$   
 $\varphi = \arccos(z/\rho)$



spherical  $\rightarrow$  cylindrical  $r^2 = \rho^2 \sin^2 \varphi$   
 $\theta = \theta$   
 $z = \rho \cos \varphi$

cylindrical  $\rightarrow$  spherical  $\rho = \sqrt{r^2 + z^2}$   
 $\theta = \theta$   
 $\varphi = \arccos(z/\rho)$

spherical coordinates useful for surfaces with a center or point of symmetry

**sphere:**  $\rho = c$       **vertical half-plane:**  $\theta = c$       **half cone:**  $\varphi = c$  ( $0 < \theta < \pi/2$ )

Example      Convert  $2x^2 + 2y^2 - z^2 = 0$       (cone)

$$2(\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta) = \rho^2 \cos^2 \varphi$$

$$2\rho^2 \sin^2 \varphi = \rho^2 \cos^2 \varphi$$

$$\tan^2 \varphi = 1/2$$

$$\varphi = \sim 35^\circ \text{ or } \sim 145^\circ$$

Example      Convert  $x^2 + y^2 + z^2 - 4z = 0$       (sphere)

$$x^2 + y^2 + z^2 = 4z$$

$$\rho^2 = 4\rho \cos \varphi \rightarrow \rho = 4 \cos \varphi$$

Example      Convert  $x^2 + y^2 = 4$       (cylinder)

$$\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta = 4$$

$$\rho^2 \sin^2 \varphi = 4$$

$$\rho \sin \varphi = 2$$

Example      Convert  $x^2 + y^2 = z$       (circular paraboloid)

$$\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta = \rho \cos \varphi$$

$$\rho^2 \sin^2 \varphi = \rho \cos \varphi$$

$$\rho \sin^2 \varphi = \cos \varphi$$

$$\rho = \cos \varphi \csc^2 \varphi \quad \text{or} \quad \rho = \cot \varphi \csc \varphi$$