


NOTES – Conics, Parametric Equations, and Polar Coordinates (Chpt 10)


10.3 Parametric equations

slope: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$ $\frac{d^3y}{dx^3} = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}}$

arc length: $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

surfaces of revolution $x = f(t)$ $y = g(t)$

$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ 

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10.4 – 10.5 Polar Coordinates (r, θ)

$x = r \cos \theta$ $y = r \sin \theta$ $\tan \theta = y/x$ $r^2 = x^2 + y^2$

tangent lines

$r = f(\theta)$ $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$

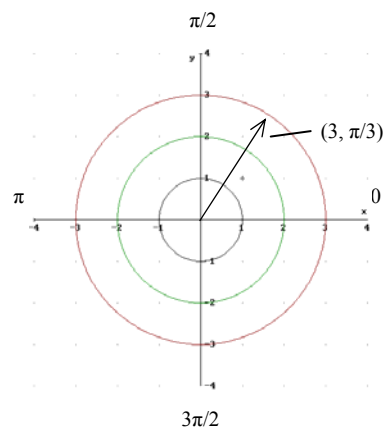
horizontal tangents: $dy/d\theta = 0$ $dx/d\theta \neq 0$

vertical tangents: $dx/d\theta = 0$ $dy/d\theta \neq 0$

at pole: If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole to $r = f(\theta)$
(tangent line goes through the pole; may be more than one)

area $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

arc length $s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$



surfaces of revolution $r = f(\theta)$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta \quad \text{about polar axis, } \theta = 0$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta \quad \text{about } \theta = \pi/2$$

$[\alpha, \beta]$ is the interval to trace the graph once, eg., $[0, \pi]$ for the circle $r = \cos \theta$, $[0, 2\pi]$ for the limaçon $r = 2 + 3 \cos \theta$

points of intersection – sketch the graphs

$$r = 1 - 2\cos\theta \quad \text{and} \quad r = 1$$

set the equations equal to each other to find intersections: when $\theta = \pi/2$ and $\theta = 3\pi/2$

third point of intersection at $(1, \pi)$ for $r = 1$ and $(-1, 0)$ for $1 - 2\cos\theta$
the paths cross, but not at the same time (no collision)

10.6 Polar Equations and Conics

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta} \quad \text{are conics; } \underline{\text{focus is at the pole}}$$

$$e = \frac{c}{a} \quad |d| = \text{distance between focus at the pole } (0, 0) \text{ and directrix}$$

$$e < 1 \rightarrow \text{ellipse} \quad e = 1 \rightarrow \text{parabola} \quad e > 1 \rightarrow \text{hyperbola}$$

To graph:

1. Solve for e and analyze the equation to orient the graph.
2. Find vertex coordinates at the appropriate angles – some r values may be negative!
3. You can solve for a and c from the graph. If b is desired, use:
 ellipse: $b^2 = a^2 - c^2 = a^2(1 - e^2)$
 hyperbola: $b^2 = c^2 - a^2 = a^2(e^2 - 1)$

To create an equation:

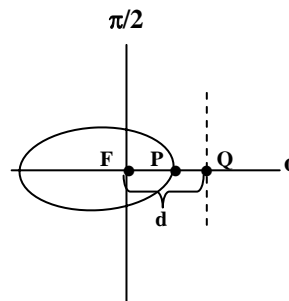
1. Sketch the graph based on information given to find the general equation.
2. Solve for a and c to find e (except for a parabola: $e = 1$). Construct an initial equation.
3. If d is not given:
 - a) plug in known θ , r , and e , and solve for d .
 - or b) calculate it:
 For a parabola, $d = 2p$; the distance from the focus to the vertex is the same as the distance from the vertex to the directrix.

To calculate d for ellipses and hyperbolas:

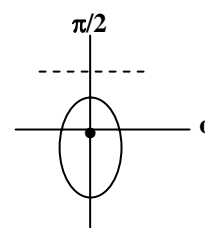
Let F = focus at pole, P = vertex, and Q = point on the directrix in a line with F and P . (Actually, this works with any point P on the curve and any point Q on the directrix.)

$e = \frac{PF}{PQ}$ If you know e and the distance from the focus to the vertex, you can calculate the distance from the vertex to the directrix.

The sum of the two distances = d .



Example: Find the polar equation for an ellipse with focus at the pole and vertices at $(2, \pi/2)$ and $(6, 3\pi/2)$.



$$r = \frac{ed}{1+e \sin \theta} \quad a = 4 \quad c = 2 \quad e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}$$

$$r = \frac{(1/2)d}{1+(1/2)\sin \theta} \quad \text{for } (2, \pi/2), \quad 2 = \frac{(1/2)d}{1+(1/2)\sin(\pi/2)} = \frac{(1/2)d}{1+1/2} = \frac{d}{3} \quad \rightarrow \quad d = 6$$

$$r = \frac{(1/2)(6)}{1+(1/2)\sin \theta} = \frac{3}{1+(1/2)\sin \theta} = \frac{6}{2+\sin \theta}$$

or:

$$r = \frac{(1/2)d}{1+(1/2)\sin \theta}$$

$$PF = 2 \quad e = \frac{1}{2} = \frac{PF}{PQ} = \frac{2}{PQ} \quad \rightarrow \quad PQ = 4 \quad d = PF + PQ = 2 + 4 = 6$$

$$r = \frac{(1/2)(6)}{1+(1/2)\sin \theta} = \frac{3}{1+(1/2)\sin \theta} = \frac{6}{2+\sin \theta}$$

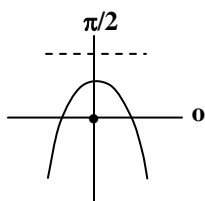
To help remember the equations, note the position of the directrix.

For the " $1 + e \sin \theta$ " equations, the directrix crosses the positive y-axis; for the " $1 - e \sin \theta$ " equations, the directrix crosses the negative y-axis.

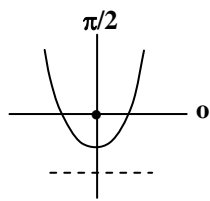
For the " $1 + e \cos \theta$ " equations, the directrix crosses the positive x-axis; for the " $1 - e \cos \theta$ " equations, the directrix crosses the negative x-axis.

*** for $d > 0$ ***

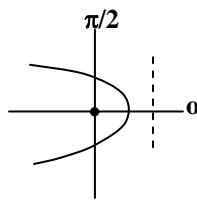
parabola $e = 1$



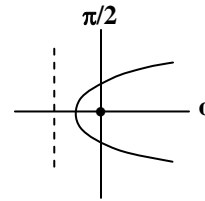
$$r = \frac{ed}{1 + e \sin \theta}$$



$$r = \frac{ed}{1 - e \sin \theta}$$

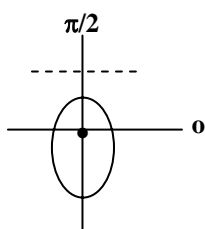


$$r = \frac{ed}{1 + e \cos \theta}$$

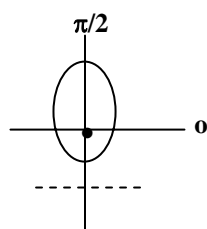


$$r = \frac{ed}{1 - e \cos \theta}$$

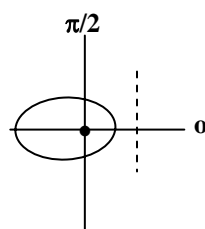
ellipse $e < 1$



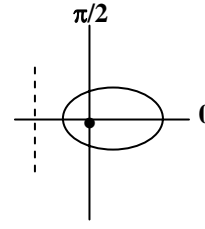
$$r = \frac{ed}{1 + e \sin \theta}$$



$$r = \frac{ed}{1 - e \sin \theta}$$

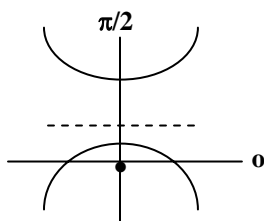


$$r = \frac{ed}{1 + e \cos \theta}$$

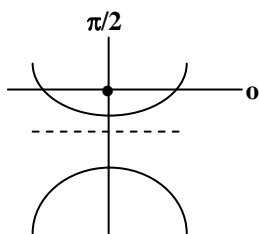


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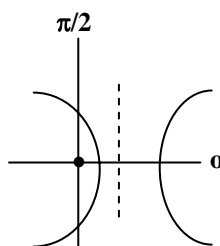
hyperbola $e > 1$



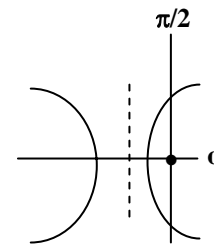
$$r = \frac{ed}{1 + e \sin \theta}$$



$$r = \frac{ed}{1 - e \sin \theta}$$



$$r = \frac{ed}{1 + e \cos \theta}$$

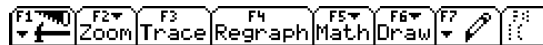


$$r = \frac{ed}{1 - e \cos \theta}$$

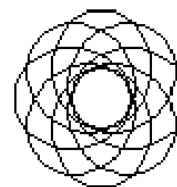
Subtracting a value from θ rotates the graph counterclockwise.

Adding a value to θ rotates the graph clockwise.

e.g., $r = \frac{3}{2 + \cos(\theta - \pi/4)}$



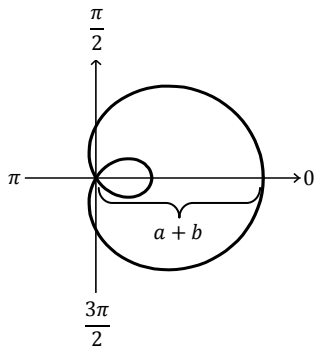
Graph made by rotating
 $r = \frac{3}{2 + \cos \theta}$
 in $\pi/4$ increments



MAIN RAD AUTO FDL

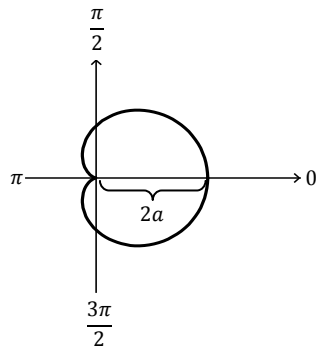
Special Polar Graphs

Limaçons: (positive cosine orientation) period = 2π $r = a \pm b \cos(\theta)$ $r = a \pm b \sin(\theta)$ ($a > 0, b > 0$)



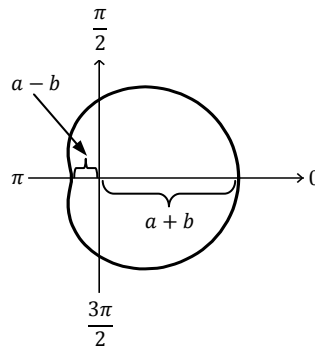
$$\frac{a}{b} < 1$$

Limaçon with inner loop



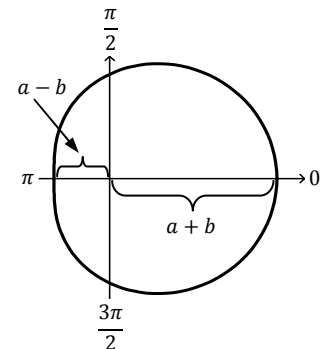
$$\frac{a}{b} = 1$$

Cardioid (heart-shaped)



$$1 < \frac{a}{b} < 2$$

Dimpled limaçon

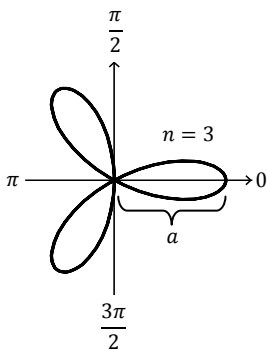


$$\frac{a}{b} \geq 2$$

Convex limaçon

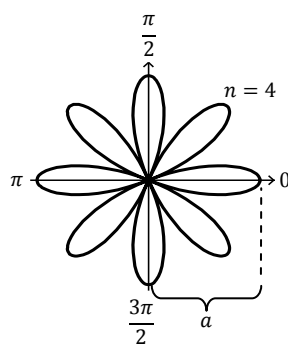
Rose Curves:

n petals if n is odd (period = π); $2n$ petals if n is even ($n \geq 2$ and period = 2π)



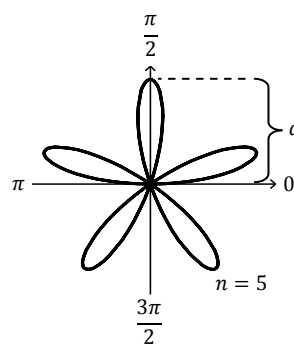
$$r = a \cos(n\theta)$$

Rose curve



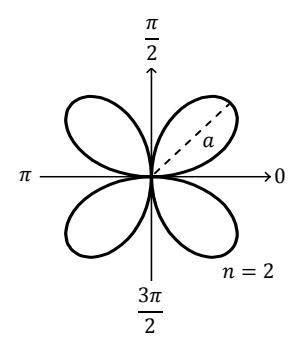
$$r = a \cos(n\theta)$$

Rose curve



$$r = a \sin(n\theta)$$

Rose curve

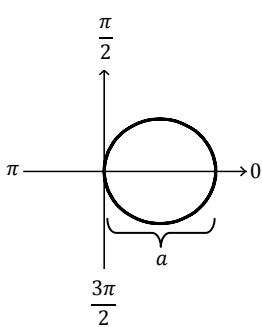


$$r = a \sin(n\theta)$$

Rose curve

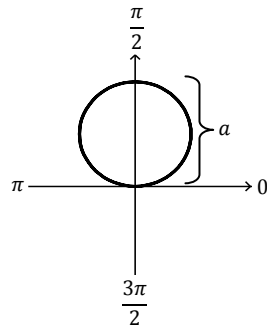
Circles and Lemniscates:

period = π



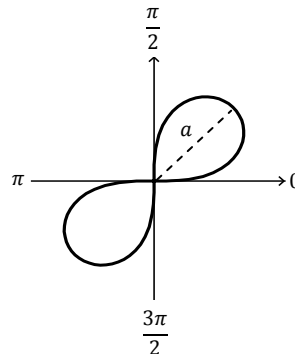
$$r = a \cos(\theta)$$

Circle



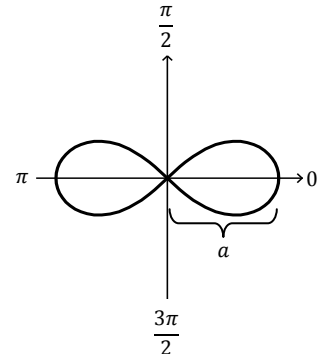
$$r = a \sin(\theta)$$

Circle



$$r^2 = a^2 \sin(2\theta)$$

Lemniscate



$$r^2 = a^2 \cos(2\theta)$$

Lemniscate