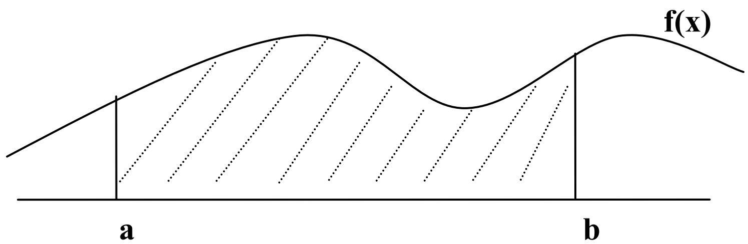
Areas by Riemann Sums

Suppose f is a continuous and non-negative function on the interval [a, b]. Suppose you are interested in finding the area under the curve between a and b.

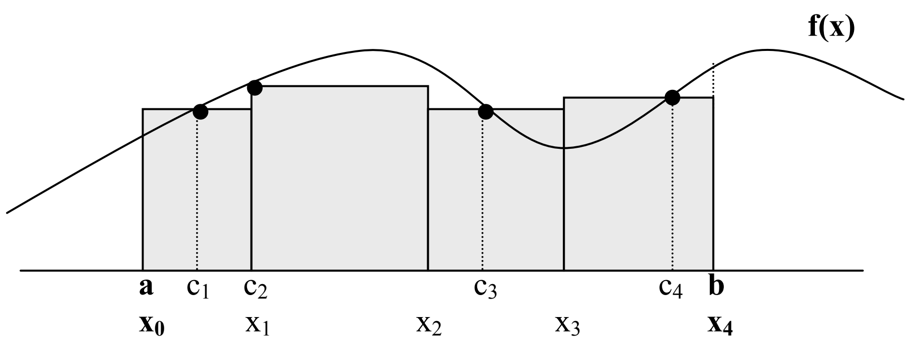


One way to estimate the area would be to divide it up into rectangles and calculate the area of each rectangle (height times width), then add the areas up. You could do it any weird way you like. We will use the notation:

$x_0, x_1, x_2, \ldots x_i, \ldots x_n$ will define the "edges" of each rectangle

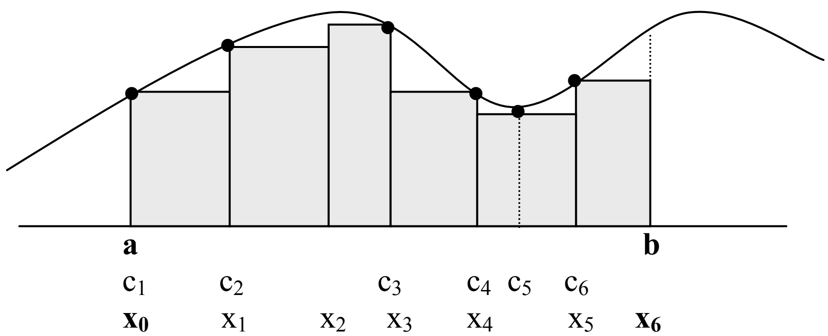
(Note that x_o must always equal a and x_n must always equal b.)

$c_1, c_2, c_3, \ldots c_i, \ldots c_n$ will denote the x value inside each rectangle where we choose to measure our rectangle height

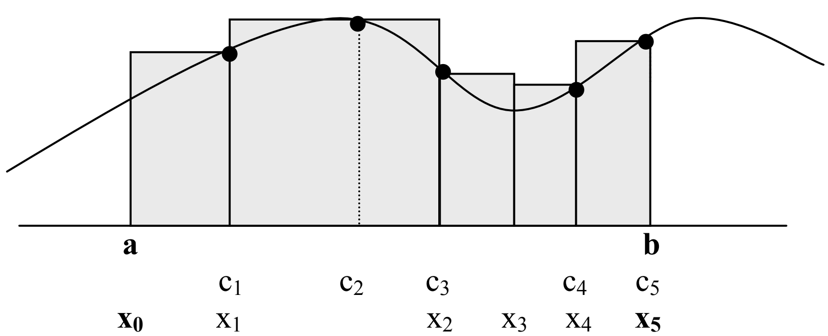


Area of first rectangle	= height • width	= $f(c_1) \cdot (x_1 - x_o)$
Area of second rectangle	= height • width	= $f(c_2) \cdot (x_2 - x_1)$
Area of third rectangle	= height • width	= $f(c_3) \cdot (x_3 - x_2)$
Area of fourth rectangle	= height • width	= $f(c_4) \cdot (x_4 - x_3)$

Choosing the values for $x_o, x_1, x_2, \ldots x_n$ is known as "partitioning" the interval [a, b]. If you choose your x's and c's so that all the rectangle heights stay **under the curve**, the sum is called a **lower sum** – it will <u>underestimate</u> the area.



If you chose your x's and c's so that all the rectangle heights stay **above the curve**, the sum is called an **upper sum** – it will <u>overestimate</u> the area.

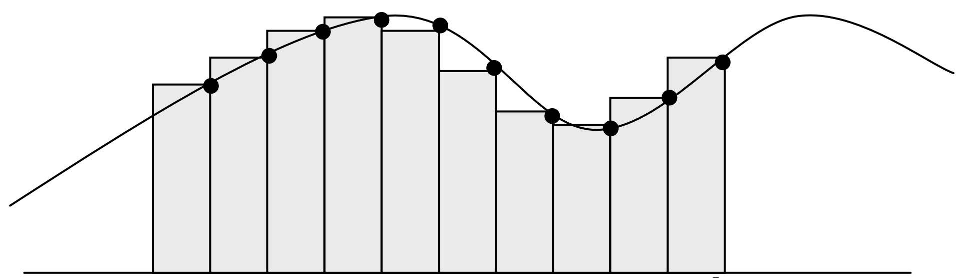


If n stands for the number of rectangles you make, then the larger n is, the better your area estimate will be, no matter how you partition the interval and no matter what values you choose for the c's. If you take the limit (as $n \to \infty$) of the sum of the rectangle areas, you will get the <u>actual</u> area under the curve (<u>not</u> an estimate).

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \bullet \Delta x_i$$

where $f(c_i)$ is the height of a particular rectangle and Δx_i is the width of that particular rectangle.

Even though this definition of area is true for any weird partition and any weird choices for c, actually finding this limit can be very difficult. If you are going to use this definition to find an area, the easiest way is to divide the interval into rectangles of **equal width** and choose **c** to be the **right endpoint** of each rectangle.



Since we have n rectangles of equal width, the width of each rectangle is $\frac{b-a}{n}$. So, in the area formula, Δx_i is always $\frac{b-a}{n}$.

Since we are choosing our c's to be the right endpoints of the rectangles:

$$c_1 = x_1 = a + 1\left(\frac{b-a}{n}\right)$$ x_1 is one width away from *a*

$$c_2 = x_2 = a + 2\left(\frac{b-a}{n}\right)$$ x_2 is two widths away from *a*

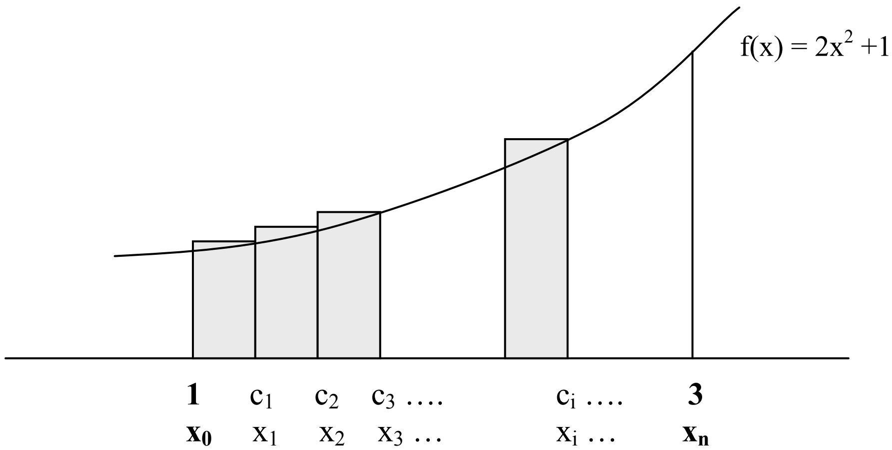
$$c_3 = x_3 = a + 3\left(\frac{b-a}{n}\right)$$ x_3 is three widths away from *a*

$$\vdots$$

$$c_i = x_i = a + i\left(\frac{b-a}{n}\right)$$ x_i is *i* widths away from *a*

Now we have a general expression for c_i that we can plug into f to find $f(c_i)$.

Example. Find the area under $f(x) = 2x^2 + 1$ on the interval [1, 3]



Each rectangle has a width: $\dfrac{b-a}{n} = \dfrac{3-1}{n} = \dfrac{2}{n} = \Delta x_i$

For each c: $c_i = x_i = a + i\left(\dfrac{b-a}{n}\right) = 1 + i\left(\dfrac{2}{n}\right) = 1 + \dfrac{2i}{n}$

Using the area definition:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \bullet \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \bullet \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(1 + \frac{2i}{n}\right)^2 + 1\right] \bullet \frac{2}{n}$$

So how do we find this limit? We'll need the help of the properties of summations and the summation formulas in your book for Σc, Σi, Σi^2, and Σi^3. Remember that for this summation, the variable is **i** and **n** is just a constant. Use summation properties to try and create simple summations that match your formulas (Σc, Σi, Σi^2, Σi^3, etc).

$$\sum_{i=1}^{n}\left[2\left(1+\frac{2i}{n}\right)^2+1\right]\cdot\frac{2}{n} \quad = \quad \frac{2}{n}\sum_{i=1}^{n}\left[2\left(1+\frac{2i}{n}\right)^2+1\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[2\left(1+\frac{2i}{n}+\frac{2i}{n}+\frac{4i^2}{n^2}\right)+1\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[2+\frac{8i}{n}+\frac{8i^2}{n^2}+1\right]$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[3+\frac{8i}{n}+\frac{8i^2}{n^2}\right]$$

$$= \frac{2}{n}\left(\sum_{i=1}^{n}3+\sum_{i=1}^{n}\frac{8i}{n}+\sum_{i=1}^{n}\frac{8i^2}{n^2}\right)$$

$$= \frac{2}{n}\left(\sum_{i=1}^{n}3+\frac{8}{n}\sum_{i=1}^{n}i+\frac{8}{n^2}\sum_{i=1}^{n}i^2\right)$$

Now, use the formulas in your book to replace $\Sigma 3$, Σi, and Σi^2 with the appropriate expressions involving *n*.

$$\frac{2}{n}\left(\sum_{i=1}^{n}3+\frac{8}{n}\sum_{i=1}^{n}i+\frac{8}{n^2}\sum_{i=1}^{n}i^2\right)$$

$$= \frac{2}{n}\left[3n+\frac{8}{n}\left(\frac{n(n+1)}{2}\right)+\frac{8}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \frac{2}{n}\cdot 3n + \frac{2}{n}\cdot\frac{8}{n}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n}\cdot\frac{8}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

At this point it is very important to <u>resist the urge to multiply these quantities out</u>! Your goal is to take the limit of this expression as $n \to \infty$. What you want to do is to rearrange your factors so that you have an n in the denominator under each factor in the numerator with an n in it:

$$\frac{2}{n} \bullet 3n + \frac{2}{n} \bullet \frac{8}{n}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n} \bullet \frac{8}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{2 \bullet 3 \bullet n}{n} + \frac{2 \bullet 8}{2}\frac{\bullet n}{\bullet n}\frac{\bullet (n+1)}{\bullet n} + \frac{2 \bullet 8}{6}\frac{\bullet n}{\bullet n}\frac{\bullet (n+1)}{\bullet n}\frac{\bullet (2n+1)}{\bullet n}$$

$$= 6 \bullet 1 + 8 \bullet 1 \bullet \left(\frac{(n+1)}{n}\right) + \frac{8}{3} \bullet 1 \bullet \frac{(n+1)}{n} \bullet \frac{(2n+1)}{n}$$

$$= 6 + 8 \bullet \left(\frac{n}{n} + \frac{1}{n}\right) + \frac{8}{3} \bullet \left(\frac{n}{n} + \frac{1}{n}\right) \bullet \left(\frac{2n}{n} + \frac{1}{n}\right)$$

$$= 6 + 8 \bullet \left(1 + \frac{1}{n}\right) + \frac{8}{3} \bullet \left(1 + \frac{1}{n}\right) \bullet \left(2 + \frac{1}{n}\right)$$

Since every $\frac{1}{n} \to 0$ as $n \to \infty$, it will now be easy to find the limit of this expression.

$$\lim_{n\to\infty}\left[6 + 8\left(1 + \frac{1}{n}\right) + \frac{8}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right]$$

$$= 6 + 8(1+0) + \frac{8}{3}(1+0)(2+0)$$

$$= 6 + 8 + \frac{8}{3}(1)(2)$$

$$= 14 + \frac{16}{3}$$

$$= \frac{58}{3}$$