APPLICATIONS OF EXTREMA OF FUNCTIONS OF TWO VARIABLES

Applications of extrema are found in many diverse fields such as economics, business, finance, geometry, and physics. Here are some examples:

1. Three numbers have a fixed sum K. What value of these numbers will make their product a maximum?

Given x + y + z = k, maximize p = xyz

Let z = k – x – y then $p = xy(k - x - y) = xyk - x^2y - xy^2$.

$p_x = yk - 2xy - y^2$, *and* $p_y = xk - x^2 - 2xy$

Setting both of these equal to zero, we have

k – y = 2x and k – x = 2y which implies that x = y, so that x = y = z = k/3.

So the maximum product is $\frac{k^3}{27}$.

2. Given the plane 3x + 4y + 5z – 10 = 0, find the point on the plane that is nearest to the origin.

Let (x,y,z) be the point on the plane. Then the distance to the origin is

$d = \sqrt{x^2 + y^2 + z^2}$. Maximizing d is the same as maximizing d^2.

Let $D = d^2 = x^2 + y^2 + z^2$. Since z is on the plane, $z = \frac{10 - 3x - 4y}{5}$

and $D = x^2 + y^2 + \frac{1}{25}(10 - 3x - 4y)^2$. Taking partial derivatives with respect to x and y gives

$$D_x = 34x + 12y - 30, \ \ and \ \ D_y = 12x + 41y - 40$$

Setting both of these equal to zero and solving for x and y yields

x = 3/5, y = 4/5, and z = 1.

By the second partials derivative test, $D_{xx} = 17, D_{yy} = 41, \ D_{xy} = 12$.

$$D_{xx} D_{yy} - (D_{xy})^2 = 553$$

This shows that we have found the minimum.

3. We wish to construct a rectangular box with a volume of 32 cubic feet. We also wish this box to require the least material. What should be the dimensions of this box if it has no top?

Let the dimensions of the box be x, y, z for the length, width and height.

The surface area is then xy + 2xz + 2yz, since there is no top.

The volume is xyz = 32, or z = 32/xy.

Writing the surface area as a function of x,y

$$A = xy + \frac{64}{y} + \frac{64}{x}$$

Taking partial derivatives,

$$A_x = y - \frac{64}{x^2}, \quad A_y = x - \frac{64}{y^2}$$

Setting both of these equal to zero and solving gives x = 4, y = 4, z = 2.

4. A rectangular solid has three sides in the coordinate planes and one vertex on the plane x + y + z = a. What is the maximum volume of the solid?

V = xyz or V = xy(a – x – y)

Taking partial derivatives

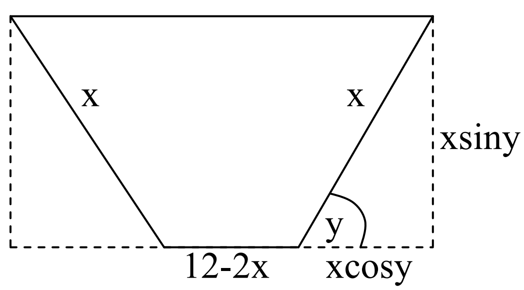
$$V_x = ay - 2xy - y^2, \quad V_y = ax - 2xy - x^2$$

Setting these equal to zero gives

a – 2y – x = 0 and a – 2x – y = 0, which shows x = y = a/3, z = a/3.

Maximum volume is $\frac{a^3}{27}$.

5. A water trough in the shape of a trapezoid is constructed from a 12 inch-wide sheet of metal by bending up the sides. What is the maximum area of a cross section?



The area of a trapezoid is one half the sum of the bases times the height.

A = 1/2[(12 – 2x) + (12 – 2x) + 2xcosy](xsiny) or

A = [(12 – 2x) + (xcosy)](xsiny)

Then taking partial derivatives

$$A_x = 12\sin y - 4x\sin y + 2x\cos y\sin y$$
$$A_y = 12x\cos y - 2x^2\cos y + x^2(2\cos^2 y - 1)$$

Setting $A_x = 0$, dividing by siny, and solving for cosy gives

$$\cos y = \frac{2x-6}{x}$$

Setting $A_y = 0$, and substituting $\frac{2x-6}{x}$ for cosy yields

$$0 = 3x^2 - 12x \Rightarrow x = 0, x = 4, \cos y = \frac{1}{2}, \sin y = \frac{\sqrt{3}}{2}, \text{ so } A = 12\sqrt{3}.$$

Exercises:

Find the critical points and relative extrema for:

1. $f(x,y) = x^2 - 2xy + 2y^2 + 4x$

2. $f(x,y) = e^{x^3+y^3}$

3. The length and girth of a rectangular package may not exceed a total of 96 inches. What is its largest volume?

4. An open rectangular box has a volume equal to 1. What dimensions will give the box the smallest surface area?

5. Find the shortest distance from the point (1,2,3) to the plane x + 3y + 5z = 6

Answers:

1. (-4,-2), -8
2. Saddle point at (0,0)
3. V = 8192
4. $l = w = 2^{\frac{1}{3}},\ h = 2^{\frac{-2}{3}}$
5. $\frac{16}{\sqrt{35}}$.