

THE SECOND-DEGREE EQUATION AND ROTATION OF AXES

The ellipse/circle, parabola, and hyperbola functions are all cases of the second-degree equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

	Ax^2	$+ Bxy$	$+ Cy^2$	$+ Dx$	$+ Ey$	$+ F$
	A	B	C	D	E	F
Circle	$= C$	$= 0$	$= A$			
Ellipse	$\neq C$ (same sign)	$= 0$	$\neq A$ (same sign)			
Parabola	$= 0$ if $C \neq 0$	$= 0$	$= 0$ if $A \neq 0$			
Hyperbola	opp. sign of C	$= 0$	opp. sign of A			
Hyperbola	$= 0$	$\neq 0$ (rotated)	$= 0$			
Center/vertex not at origin				$A \neq 0$ and $D \neq 0$ &/or $C \neq 0$ and $E \neq 0$		
Rotated axis		$\neq 0$				

The **discriminant** $B^2 - 4AC$ can be used to classify the conic section (except in degenerate cases):

$$B^2 - 4AC < 0 \rightarrow \text{ellipse or circle}$$

$$B^2 - 4AC = 0 \rightarrow \text{parabola}$$

$$B^2 - 4AC > 0 \rightarrow \text{hyperbola}$$

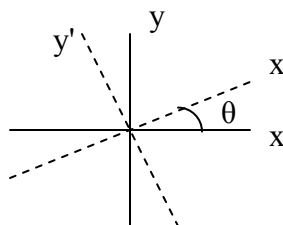
If there is an xy term in the equation, then the axes of the conic are rotated so that they are not parallel to the x -axis or the y -axis. To obtain a **standard form** of the conic equation, we use **rotation of axes** to eliminate the xy term. This procedure rotates the axes through an **acute** angle θ in a counterclockwise direction ($0 < \theta < \pi/2$). The new axes are labeled x' and y' .

θ = angle of rotation from x -axis

$$\cot 2\theta = \frac{A - C}{B}$$

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$



Substitution for x and y in the original equation will eliminate the xy term. The new equation will be in the form of $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$. The F term is invariant and is repeated in the new equation as F' .

Example 1. Rotate the axes to eliminate the xy term: $x^2 - 10xy + y^2 + 1 = 0$

$$A = 1 \quad B = -10 \quad C = 1 \quad F = 1 = F'$$

$$B^2 - 4AC = 100 - 4 = 96 > 0 \quad \rightarrow \quad \text{hyperbola}$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{0}{-10} \quad \cos 2\theta = 0 \quad \rightarrow \quad 2\theta = \pi/2 \quad \rightarrow \quad \theta = \pi/4$$

$$\cos \theta = \sqrt{2}/2 \quad \sin \theta = \sqrt{2}/2$$

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \quad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

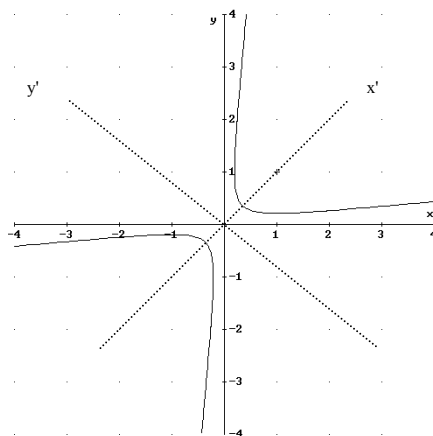
Substitute into the original equation:

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 - 10\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 + 1 = 0$$

$$4x'^2 - 6y'^2 = 1$$

$$\frac{x'^2}{(1/4)} - \frac{y'^2}{(1/6)} = 1$$

This is a hyperbola rotated through 45° .
 x' is the transverse axis and the center is at the origin ($D' = E' = 0$).



Example 2. Rotate the axes to eliminate the xy term: $40x^2 + 36xy + 25y^2 - 52 = 0$

$$A = 40 \quad B = 36 \quad C = 25 \quad F = -52 = F'$$

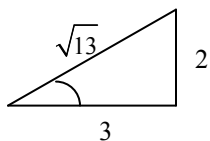
$$B^2 - 4AC = 36^2 - 4(40)(25) = -2704 < 0 \quad \rightarrow \quad \text{ellipse or circle}$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{5}{12} \quad \text{Use an identity: } \cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta} = \frac{5}{12}$$

$$\text{Solve for } \theta: \quad (3 \cot \theta + 2)(2 \cot \theta - 3) = 0$$

$$\cot \theta = 3/2 \quad (\text{use only the positive solution; } \theta \text{ is in Quadrant I})$$

$$\theta \approx 33.7^\circ$$



$$\cos \theta = \frac{3}{\sqrt{13}}$$

$$\sin \theta = \frac{2}{\sqrt{13}}$$

$$x = \frac{3}{\sqrt{13}}x' - \frac{2}{\sqrt{13}}y'$$

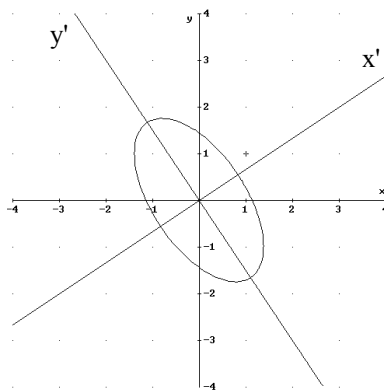
$$y = \frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y'$$

Substitute into the original equation:

$$40\left(\frac{3}{\sqrt{13}}x' - \frac{2}{\sqrt{13}}y'\right)^2 + 36\left(\frac{3}{\sqrt{13}}x' - \frac{2}{\sqrt{13}}y'\right)\left(\frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y'\right) + 25\left(\frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y'\right)^2 - 52 = 0$$

$$\frac{676}{13}x'^2 + \frac{169}{13}y'^2 = 52$$

$$x'^2 + \frac{y'^2}{52/13} = 1$$



This ellipse is rotated through $\sim 33.7^\circ$.
The major axis is along the y' axis, and
the center is at the origin ($D' = E' = 0$).

[Note: finding θ can also be done using the equation $\tan 2\theta = \frac{B}{A-C} = \frac{12}{5}$

and the identity $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$.]

Example 3. Rotate the axes to eliminate the xy term: $x^2 + 2xy + y^2 - 2x + 2y = 0$

$$A = 1 \quad B = 2 \quad C = 1 \quad F = 0 = F'$$

$$B^2 - 4AC = 4 - 4 = 0 \quad \rightarrow \quad \text{parabola}$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{0}{2}$$

$$\cos 2\theta = 0 \quad \rightarrow \quad 2\theta = \pi/2 \quad \rightarrow \quad \theta = \pi/4$$

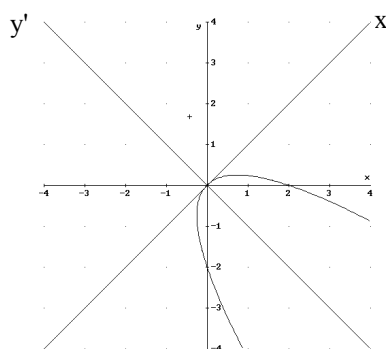
$$\cos \theta = \sqrt{2}/2 \quad \sin \theta = \sqrt{2}/2$$

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' \quad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

Substitute into the original equation:

$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 + 2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 - 2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right) + 2\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) = 0$$

$$2x'^2 + 2\sqrt{2}y' = 0 \quad x'^2 = -\sqrt{2}y'$$



This parabola opens downward along the y' axis, which is rotated 45° counterclockwise. The vertex is at the origin.

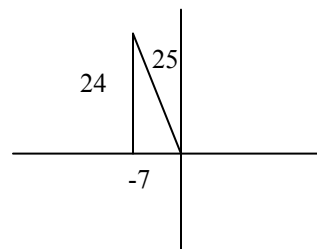
If the cotangent (or tangent) value is negative, then 2θ is an obtuse angle in Quadrant II. In QII, $\cos 2\theta$ is negative and $\sin 2\theta$ is positive.

Example 4. Rotate the axes to eliminate the xy term: $16x^2 - 24xy + 9y^2 - 60x - 80y + 400 = 0$

$$A = 16 \quad B = -24 \quad C = 9 \quad F = 400 = F'$$

$$B^2 - 4AC = 24^2 - (4)(16)(9) = 0 \rightarrow \text{parabola}$$

$$\cot 2\theta = \frac{A-C}{B} = -\frac{7}{24}$$



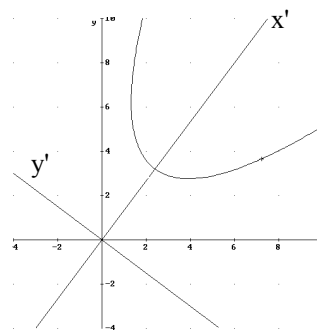
You can use various identities to find $\sin \theta$ and $\cos \theta$, for example:

$$\begin{aligned} \cos 2\theta = -7/25 & \quad \cos 2\theta = 2\cos^2\theta - 1 \rightarrow \cos \theta = 3/5 \\ \sin 2\theta = 24/25 & \quad \sin 2\theta = 2\sin\theta\cos\theta \rightarrow \sin \theta = 4/5 \end{aligned}$$

$$x = \frac{3x' - 4y'}{5} \quad y = \frac{4x' + 3y'}{5} \quad \theta \approx 53.1^\circ$$

Substitute into the original equation:

$$16\left(\frac{3x' - 4y'}{5}\right)^2 - 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 9\left(\frac{4x' + 3y'}{5}\right)^2 - 60\left(\frac{3x' - 4y'}{5}\right) - 80\left(\frac{4x' + 3y'}{5}\right) + 400 = 0$$



$$25y'^2 - 100x' + 400 = 0 \rightarrow y'^2 - 4x' + 16 = 0 \rightarrow y'^2 = 4(x' - 4)$$

This is a parabola with its axis along the x' axis and vertex at $(4, 0)$ in the $x'y'$ system.