Bernoulli Goes to the Bank

Introduction

A question commonly asked by those students struggling with a required mathematics course is, “What is this stuff good for?” Though asked in every mathematics course that I have taught, I think business calculus is the one course where this question requires the strongest response. For in my other classes—pre-algebra, algebra, etc.—I can argue that one is learning a universal language of quantification. Subsequently, to essentially ask ‘of what good is this algebraic language?’ is to miss the whole point of having available a new, powerful, and exact means of communication. To not have this communication means at my disposal could be likened to not being able to speak English in a primarily English-speaking country. To say that this would be a handicap definitely is an understatement! Yet this is precisely what happens when one doesn’t speak mathematics in a technological world bubbling over with mathematical language: e.g. numbers, data, charts, and formulas. I have found through experience that the previous argument makes a good case for pre-algebra and algebra; however, making a similar case for business calculus may require more specifics in a day when EXCEL rules. In this article, we will explore one very essential specific in the modern world of finance, namely the growth and decay of money by the use of **differential equations**, one of the last topics encountered in a standard business calculus course.

Jacob Bernoulli’s Differential Equation

Jacob Bernoulli (1654-1705) was nestled in between the lifetimes of Leibniz and Newton, the two co-founders of calculus. Jacob was about 10 years younger than either of these men and continued the tradition of ‘standing on the shoulders of giants’.

One of Jacob's greatest contributions to mathematics *and physics* was made in the year 1696 when he found a solution to the differential equation below, which bears his name.

$$\frac{dy}{dx} = f(x)y + g(x)y^n$$

Of particular interest in this article is the case for $n = 0$:

$$\frac{dy}{dx} = f(x)y + g(x).$$

The solution is obtained via Bernoulli's 300-year-old methodology as follows.

Step1: Let $F(x)$ be such that $F'(x) = -f(x)$

Step 2: Formulate the integrating factor $e^{F(x)}$

Step 3: Multiply both sides of $\frac{dy}{dx} = f(x)y + g(x)$ by $e^{F(x)}$ to obtain

$$e^{F(x)}\left[\frac{dy}{dx}\right] = e^{F(x)}\left[f(x)y + g(x)\right] \Rightarrow$$
$$e^{F(x)}\left[\frac{dy}{dy}\right] + e^{F(x)}[-f(x)]y = e^{F(x)} \cdot g(x)$$

Where the left-hand side of the last equality is the derivative of a product

$$e^{F(x)}\left[\frac{dy}{dy}\right] + e^{F(x)}[-f(x)]y = e^{F(x)} \cdot g(x) = \frac{d}{dx}\left[e^{F(x)} \cdot y\right].$$

Step 4: To complete the solution, perform the indefinite integration.

$$\frac{d}{dx}\left[e^{F(x)} \cdot y\right] = e^{F(x)} \cdot g(x) \Rightarrow$$
$$e^{F(x)} \cdot y = \int e^{F(x)} \cdot g(x)dx + C \Rightarrow$$
$$y = y(x) = e^{-F(x)} \cdot \left[\int e^{F(x)} \cdot g(x)dx\right] + Ce^{-F(x)} \therefore$$

Admittedly, the final equality is a rather atrocious looking expression, but Bernoulli's approach will always give us the right solution if we faithfully follow the process embodied in the formula.

Velocity and Interest Rate

To be able to apply Bernoulli's differential equation to banks, banking, and money, we will first need to understand the closely connected concepts of *interest* and *interest rate*. Interest affects just about every adult in America. If you are independent, own a car or a home or both, or have a credit card or two, you probably pay or have paid interest. So, what exactly is interest? *Interest is a rental fee for the use of money. As such, an interest charge for the use of money accumulates with the passage of time just as a rental fee for the use of housing accumulates with the passage of time.* The interest charge is normally stated in terms of a *percentage interest rate*, which is multiplied by the amount borrowed in order to calculate the actual interest fee. Analogous to velocity, a time-rate of distance accumulation (e.g. $60\frac{miles}{hour}$), percentage interest rate is a time-rate of percent accumulation (e.g. $8\frac{\%}{year}$). When driving in America, the customary units of velocity are *miles per hour*. Likewise, the customary units for interest rate are *percent per year*, which we will adhere to throughout this paper. However, one should be aware that other than customary units may be used in certain situations. For example, in space travel $7\frac{miles}{\sec}$ is used to describe escape velocity from planet earth; and, when computing a credit-card charge, a monthly interest rate of $1.5\frac{\%}{month}$ may be used. Both velocity and percentage interest rate need to be multiplied by time—specified in matching units—in order to obtain the total amount accumulated, either miles or percent, as illustrated below.

On the road: $D = 75\frac{miles}{hour} \cdot 2\frac{1}{3} hours = 175 miles$

In the bank: $\% = 2\frac{percent}{month} \cdot 3\frac{1}{2} months = 7\, percent$ $

With these preliminaries out of the way, we are ready to explore how a given amount of money, called the principle (denoted by p) changes with the passing of time (denoted by t). To start, we will explore how an intrinsic differential change in time dt induces a differential change dP.

Time, Money, and Differential Change

Everyone will agree that a fixed amount of money p will change with time. Even though $p = \$10{,}000.00$ is stuffed under a mattress for twenty years in the hopes of preserving its value, the passage of twenty years will change p into something less due to the ever-present action of inflation (denoted by i in this article), which can be thought of as a negative interest rate. So properly, $p = p(t)$ where t is the independent variable and p is the dependent variable.

Let dt be a differential increment of time. Since $p = p(t)$, dt will induce a corresponding differential change dp in p via a first-order linear expression linking dp to dt:

$$dp = Kdt \Rightarrow dp = K(t)dt.$$

The exact form of the proportionality expression $K(t)$ will depend on whether principle is growing, decaying, or whether there is a number of complementary and/or competing monetary-change mechanisms at work. Any one of these mechanisms may be time dependent in and of itself necessitating the writing of K as $K = K(t)$.

The simplest case is the monetary growth mechanism where $K = rp_0$, the product of a constant interest rate r and the initial principle p_0. This implies a constant rate of dollar increase with time for a given p_0, which is the traditional simple-interest growth mechanism. Thus

$$dp = rp_0 dt : p(0) = p_0.$$

The preceding is a first-order linear differential equation written in separated form with stated initial condition. It can be easily solved in three steps:

$$\begin{aligned} &\overset{1}{\mapsto} : p(t) = p_0 rt + C \\ &\overset{2}{\mapsto} : p(0) = p_0 \Rightarrow C = p_0 \\ &\overset{3}{\mapsto} : p(t) = p_0 rt + p_0 = p_0(1 + rt) \end{aligned}.$$

One might recognize the last expression as the functional form of the *simple interest formula*.

The same differential equation can be written as

$$\frac{dp}{dt} = rp_0 : p(0) = p_0 \text{ after division by } dt .$$

This form highlights the differential-based definition of the first derivative. In words it states that *the ratio of an induced differential change of principle with respect to a corresponding, intrinsic differential change in time is constant, being equal to the applied constant interest rate times the initial principle, also constant.* Simple examination of both sides of the above differential equation reveals common and consistent units for both sides with

$$\frac{dp}{dt} \equiv \frac{dollars}{year} \;\&\; rp_0 \equiv \frac{dollars}{year} .$$

Furthermore, the reader will recognize the expression

$$\frac{dp}{dt} \equiv p'(t)$$

as the Leibniz form of the first derivative, equal to the instantaneous change of principle with respect to time—which one could immediately liken to an instantaneous "velocity" of money growth.

Bernoulli and Money

Returning to $dp = K(t)dt$, we have for the general case that $K(t) = r(t) \cdot p(t) + d(t)$ where $r(t)$ is a time-varying (variable) interest rate, $p(t)$ is the principle *currently present*, and $d(t)$ is an independent variable deposit rate. Substituting into $dp = K(t)dt$ gives

$$dp = [r(t) \cdot p(t) + d(t)]dt : p(0) = p_0 \text{ or}$$

$$\frac{dp}{dt} = r(t) \cdot p(t) + d(t) : p(0) = p_0$$

where the amount of principle present at process onset is given by $p(0) = p_0$.

Translating the into words, *the instantaneous rate of change of principle with respect to time equals the sum of two independently acting quantities: 1) the product of the variable interest rate with the principle concurrently present and 2) a variable direct-addition rate.* The preceding differential equation is applicable in the business world if the principle p is continuously growing (or declining) with time. When the interest rate is fixed $r(t) \equiv r_0$ and the independent direct-addition rate is zero $d(t) \equiv 0$, the differential equation reduces to

$$\frac{dp}{dt} = r_0 p : p(0) = p_0 .$$

Solving using separation of variables gives

$$\overset{1}{\mapsto} : \frac{dp}{p} = r_0 dt$$
$$\overset{2}{\mapsto} : \ln(p) = r_0 t + C \Rightarrow p(t) = e^C e^{r_0 t} .$$
$$\overset{3}{\mapsto} : p(0) = p_0 \Rightarrow p(t) = p_0 e^{r_0 t}$$

The final expression $p(t) = p_0 e^{r_0 t}$ is the familiar Continuous-Interest Formula for principle growth given a starting principle p_0 and constant interest rate r_0. Returning to the general differential equation

$$\frac{dp}{dt} = r(t) \cdot p(t) + d(t) : p(0) = p_0 ,$$

we see that it is Bernoulli in form with the solution given again by an atrocious expression

$$F(t) = -\int r(t) dt$$
$$p(t) = e^{-F(t)} \cdot \left[\int e^{F(t)} \cdot d(t) dt \right] + Ce^{-F(t)}$$

Upon comparison with the general solution developed in detail earlier. The initial condition $p(0) = p_0$ will be applied on a case-by-case basis as we explore the various and powerful uses of the above solution in the world of finance.

Depending on the complexity of $r(t)$ and $d(t)$, the coupled solution

$$F(t) = -\int r(t)dt$$
$$p(t) = e^{-F(t)} \cdot \left[\int e^{F(t)} \cdot d(t)dt\right] + Ce^{-F(t)} : p(0) = p_0$$

may or may not be expressible in terms of a simple algebraic expression.. Thus, since interest rates are unpredictable and out of any one individual's control (I have seen double-digit swings in both savings-account rates and mortgage rates in my lifetime), we will assume for the purpose of predictive analysis that the interest rate is constant throughout the time interval of interest $r(t) \equiv r_0$. This immediately leads to

$$p(t) = e^{r_0 t} \cdot \left[\int e^{-r_0 t} \cdot d(t)dt\right] + Ce^{r_0 t} : p(0) = p_0,$$ a considerable simplification.

The last result is our starting point for concrete applications in investment planning, mortgage analysis, and annuity planning.

Growing a Nest Egg

Case 1: If $d(t) \equiv d_0$, a constant annual deposit rate, then the last expression for $p(t)$ further simplifies to

$$p(t) = d_0 e^{r_0 t}\left[\int e^{-r_0 t} dt\right] + Ce^{r_0 t} : p(0) = p_0.$$

This can be easily solved to give

$$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t} - 1\right]$$ after applying the boundary condition $p(0) = p_0$.

Notice that the above expression consists of two distinct terms. The term $p_0 e^{r_0 t}$ corresponds to the principle accrued in an continuous interest-bearing account over a time period t at a constant interest rate r_0 given an initial lump-sum investment p_0.

Likewise, the term $\frac{d_0}{r_0}\left[e^{r_0 t}-1\right]$ results from direct principle addition via annual metered contributions into the same interest-bearing account. If either of the constants p_0 or d_0 is zero, then the corresponding term drops away from the overall expression. The following two-stage investment problem illustrates the use of

$$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t}-1\right].$$

Example 1: You inherit $\$12{,}000.00$ at age 25 and immediately invest $\$10{,}000.00$ in a corporate-bond fund paying $6\frac{\%}{year}$. Five years later, you roll this account over into a solid stock fund (whose fifty-year average is $8\frac{\%}{year}$) and start contributing $\$3000.00$ annually. **A)** Assuming continuous and steady interest, how much is this investment worth at age 68? **B)** What percent of the final total was generated by the initial $\$10{,}000.00$?

A) In the first five years, the only growth mechanism in play is that induced by the initial investment of $\$10{,}000.00$. Thus, the amount at the of the first five years is given by

$$p(5) = \$10{,}000.00e^{0.06(5)} = \$13{,}498.58.$$

The output from Stage 1 is now input to Stage 2 where both growth mechanisms act for an additional 38 years.

$$p(38) = 13{,}498.58e^{0.08(38)} + \frac{3000}{0.08}(e^{0.08(38)}-1) \Rightarrow$$
$$p(38) = \$148{,}797.22 + \$375{,}869.11 \Rightarrow$$
$$p(38) = \$528{,}666.34$$

B) The % of the final total accrued by the initial $\$10{,}000.00$ is

$$\frac{\$148{,}792.22}{\$528{,}666.34} = .281 = 28.1\%$$

Notice that the initial investment of $\$10,000.00$ *is generating* 28.1% *of the final value even though it represents only* 8% *of the overall investment of* $\$124,000.00$. *The earlier a large sum of money is inherited or received by an individual, the wiser it needs to be invested; and the more it counts later in life.*

Holding the annual contribution rate to $\$3000.00$ over a period of 38 years is not a realistic thing to do. As income grows, the corresponding annual retirement contribution should also grow. One mathematical model for this is

$$\frac{dp}{dt} = r_0 p + d_0 e^{\alpha t} : p(0) = p_0$$

where the constant annual contribution rate d_0 in the previous model d_0 has been replaced with the expression $d_0 e^{\alpha_0 t}$, allowing the annual contribution rate to be continuously compounded over a time period t at an average annual growth rate α_0. The above equation is yet another example of a solvable Bernoulli-in-form differential equation per the sequence

$$\begin{aligned}
p(t) &= e^{r_0 t} \cdot \left[\int e^{-r_0 t} \cdot d_0 e^{\alpha_0 t} dt\right] + Ce^{r_0 t} : p(0) = p_0 \Rightarrow \\
p(t) &= d_0 e^{r_0 t} \cdot \left[\int e^{(\alpha_0 - r_0)t} \cdot dt\right] + Ce^{r_0 t} : p(0) = p_0 \Rightarrow . \\
p(t) &= p_o e^{r_0 t} + \frac{d_0}{r_0 - \alpha_0}\left[e^{r_0 t} - e^{\alpha_0 t}\right] \therefore
\end{aligned}$$

Example 2: Repeat Example 1 using the annual contribution model $d(t) = 3000e^{0.03t}$.

A) Stage 1 remains the same with $p(5) = \$13,498.58$. The Stage 2 calculation now becomes

$$\begin{aligned}
p(38) &= 13,498.58e^{0.08(38)} + \frac{3000}{0.08 - .03}(e^{0.08(38)} - e^{0.03(38)}) \Rightarrow \\
p(38) &= \$148,797.22 + \$1,066,708.49 \Rightarrow \\
P(38) &= \$1,215,500.71
\end{aligned}$$

The final annual contribution is $\$3000.00e^{0.03(38)} = \$9380.31$ with the total contribution throughout the 38 years is given by

$$\int_0^{38} \$3000.00e^{0.03t}\,dt = \$100{,}000.00e^{0.03t}\,\Big|_0^{38} = \$212{,}676.83\,.$$

B) The % of the final total accrued by the initial $\$10{,}000.00$ is

$$\frac{\$148{,}792.22}{\$1{,}215{,}500.71} = .122 = 12.2\%$$

Most of us don't receive a large amount of money early in our lives. That is the reason we are a nation primarily made up of middle-class individuals. So with this in mind, we will forgo the early inheritance in our next example.

Example 3: Assume we start our investment program at age 25 with an annual contribution of $\$3000.00$ grown at a rate of $\alpha_0 = 5\%$ per year. Also assume an aggressive annual interest rate of $r_0 = 10\%$ (experts tell us that this is still doable in the long term through smart investing). **A**) How much is our nest egg worth at age 68? **B**) How does an assumed average annual inflation rate of 3% throughout the same time period alter the final result?

A) Direct substitution gives

$$p(43) = \frac{3000}{0.10 - 0.05}\left(e^{0.10(43)} - e^{0.05(43)}\right) \Rightarrow$$
$$p(43) = \$3{,}906{,}896.11$$

B) Inflation is nothing more than a negative growth rate (or interest rate) that debits the given rate. For a 3% average annual inflation rate, the true interest r_{T0} and income growth rates α_{T0} are given by the two expressions

$$r_{T0} = r_0 - i_0 = 10\% - 3\% = 7\% = 0.07$$
$$\alpha_{T0} = \alpha_0 - i_0 = 5\% - 3\% = 2\% = 0.02$$

Sadly, our true value after 43 years in terms of today's buying power is

$$p(43) = \frac{3000}{0.07 - 0.02}\left(e^{0.07(43)} - e^{0.02(43)}\right) \Rightarrow$$
$$p(43) = \$1{,}075{,}454.35$$

Paying for the Nest

Both mortgage loans and annuities are, in actuality, investment plans in reverse where one starts with a given amount of principle $p(0) = p_0$ and chips away at this initial amount until that point in time T when $p(T) = 0$. The governing equation for the case where the interest rate r_0 is fixed throughout the amortization period T is

$p(t) = p_0 e^{r_0 t} + \frac{d_0}{r_0}\left[e^{r_0 t} - 1\right]$ where d_0 now becomes the required annual payment.

Applying the condition $p(T) = 0$ leads to

$$d_0 = \frac{r_0 p_0 e^{r_0 t}}{e^{r_0 t} - 1}.$$

The fixed monthly payment m_0 is given by

$$m_0 = \frac{d_0}{12} = \frac{r_0 p_0 e^{r_0 t}}{12\left\{e^{r_0 t} - 1\right\}}$$

The continuous-interest-principle-reduction model does an excellent job of calculating nearly-correct payments when the number of compounding or principle recalculation periods exceeds four per year. Below are three other mortgage-payment formulas based on the continuous-interest model that we will leave to the reader to verify.

1. First Month's Interest: $\frac{r_0 p_0}{12}$

2. Total Interest I Payment : $I = p_0\left[\frac{r_0 T e^{r_0 T}}{e^{r_0 T} - 1} - 1\right]$

3. Total Amount Paid $A = p_0 + I$: $A = \frac{r_0 T p_0 e^{r_0 T}}{e^{r_0 T} - 1}$

Example 4: $\$250{,}000.00$ is borrowed for 30 years at 5.75%. Calculate the monthly payment, total repayment , and total interest repayment assuming no early payout.

$$m_0 = \frac{0.0575(\$250{,}000.00)e^{0.0575(30)}}{12(e^{0.0575(30)} - 1)} = \$1457.62$$

$$A = \frac{0.0575(30)(\$250{,}000.00)e^{0.0575(30)}}{(e^{0.0575(30)} - 1)} = \$524{,}745.50$$

$$I = A - p_0 = \$524{,}745.50 - \$250{,}000.00 = \$274{,}745.51$$

Many people justify an initially-high mortgage payment due to the fact that ‘the mortgage is being paid off in cheaper dollars.” This statement refers to the effects of inflation on future mortgage payments. Future mortgage payments are simply not worth as much in today’s terms as current mortgage payments. In fact, if we project t years into the loan and the continuous annual inflation rate has been i_0 throughout that time period, then the present value of our future payment m_{PV} is

$$m_{PV} = \frac{r_0 P_0 e^{r_0 T}}{12(e^{r_0 T} - 1)} e^{-i_0 t} .$$

To illustrate, in **Example 4** the present value of a payment made 21 years from now, assuming $i_0 = 3 \frac{\%}{year}$ is $m_{PV} = \$1457.62e^{-0.03(21)} = \$776.31$. Thus, under stable economic conditions, our ability to comfortably *afford* the mortgage should increase over time. This is a case where inflation actually works in our favor. Continuing with this discussion, if we are paying off our mortgage with cheaper dollars, then what is the present value of the total amount paid A_{PV}? A simple definite integral—interpreted as continuous summing—provides the answer

$$A_{PV} = \int_0^T \left[\frac{r_0 P_0 e^{r_0 T}}{e^{r_0 T} - 1} \right] e^{-i_0 t} dt = \frac{r_0 P_0 (e^{r_0 T} - e^{(r_0 - i_0)T})}{i_0 (e^{r_0 T} - 1)}$$

Returning again to Example 4, the present value of the total 30-year repayment stream is $A_{PV} = \$345{,}999.90$.

Example 5: Compare m_0, A, and A_{PV} for a mortgage where $p_0 = \$300{,}000.00$ if the fixed interest rates are: $r_{30years} = 6\%$, $r_{20years} = 5.75\%$, and $r_{15years} = 5.0\%$. Assume a steady annual inflation rate of $i_0 = 3\%$ and no early payout. In this example, we dispense with the calculations and present the results in Table 1**.**

FIXED RATE MORTGAGE WITH $P_0 = \$300{,}000.00$				
Terms	*r*	*M*	*A*	A_{PV}
$T = 30$	6.00%	$1797.05	$646,938.00	$426,569.60
$T = 20$	5.75%	$2103.57	$504,856.80	$379.642.52
$T = 15$	5.00%	$2369.09	$426,436.20	$343,396.61

Table 1: Fixed Rate Mortgage Comparison

Table 1 definitely shows the mixed advantages/disadvantages of choosing a short-term or long-term mortgage. For a fixed principle, long-term mortgages have lower monthly payments. They also have a much higher overall repayment, although the total repayment is dramatically reduced by the inflation factor. The mortgage decision is very much an individual one and should be done considering all the facts within the scope of the broader economic picture.

Example 6: Our last example in this article is an annuity problem. Annuities are simply mortgages in reverse where monthly payouts are made, instead of monthly payments, until the principle is reduced to zero.

You retire at age 68 and invest money earned via **Example 3** in an annuity paying $4.5\frac{\%}{year}$ to be amortized by age 90. What is the monthly payout to you in today's terms? The phrase, 'in today's terms", means we let $p_0 = p_{PV} = \$1{,}075{,}454.35$. Thus,

$$m_0 = \frac{(0.045)(\$1{,}075{,}454.35)e^{(0.045)24}}{12(e^{(0.045)24} - 1)} = \$6{,}106.79 .$$

The monthly income provided by the annuity looks very reasonable referencing to the year 2005. But, unfortunately, it is a fixed-income annuity that will continue as fixed for 24 years. And, what happens during that time? Inflation! To calculate the present value of that monthly payment, say at age 84, our now well-known inflation factor is used to obtain

$$m_0 = \$6{,}106.79e^{-.03(16)} = \$3778.80 .$$

Conclusion

The power provided by the techniques in this short section on finance is nothing short of miraculous. We have used Bernoulli-in-form differential equations to model and solve problems in inflation, investment planning, and installment payment determination (whether loans or annuities). We have also revised the interpretation of the definite integral as a continuous sum in order to obtain the present value of a total repayment stream many years into the future. These economic and personal issues are very much today's issues, and calculus still very much remains a worthwhile tool-of-choice (even for mundane earthbound problems) some 300 years after its inception.