Simultaneous Determination of Two Laplace Transforms

Problem: Find $L[t\cos(at)]$ and $L[t\sin(at)]$

Step 1: By definition, $L[t\cos(at)] = \int_0^\infty e^{-st}[t\cos(at)]dt$, $L[t\sin(at)] = \int_0^\infty e^{-st}[t\sin(at)]dt$

Step 2: Integrate $\int_0^\infty e^{-st}[t\cos(at)]dt$ by parts using the standard parts formula $\int_a^b udv = uv|_a^b - \int_a^b vdu$. Let $u = t\cos(at), dv = e^{-st}$ and proceed for just one iteration.

$$\int_0^\infty e^{-st}[t\cos(at)]dt = \frac{e^{-st}}{-s}t\cos(at)|_0^\infty - \left\{\frac{-1}{s}\int_0^\infty e^{-st}\left(\cos(at) - at\sin(at)\right)dt\right\} \Rightarrow$$

$$\int_0^\infty e^{-st}[t\cos(at)]dt = 0 + \frac{1}{s}\int_0^\infty e^{-st}\cos(at)dt - \frac{a}{s}\int_0^\infty e^{-st}t\sin(at)dt \Rightarrow$$

$$L[t\cos(at)] + \frac{a}{s}L[t\sin(at)] = \frac{1}{s}L[\cos(at)]$$

Step 3: Likewise, integrate $\int_0^\infty e^{-st}[t\sin(at)]dt$ by parts using the standard parts formula $\int_a^b udv = uv|_a^b - \int_a^b vdu$. Let $u = t\sin(at), dv = e^{-st}$ and proceed for just one iteration.

$$\int_0^\infty e^{-st}[t\sin(at)]dt = \frac{e^{-st}}{-s}t\sin(at)|_0^\infty - \left\{\frac{-1}{s}\int_0^\infty e^{-st}\left(\sin(at) + at\cos(at)\right)dt\right\} \Rightarrow$$

$$\int_0^\infty e^{-st}[t\sin(at)]dt = 0 + \frac{1}{s}\int_0^\infty e^{-st}\sin(at)dt + \frac{a}{s}\int_0^\infty e^{-st}t\cos(at)dt \Rightarrow$$

$$\frac{-a}{s}L[t\cos(at)] + L[t\sin(at)] = \frac{1}{s}L[\sin(at)]$$

Step 4: Set up the system of equations for $L[t\cos(at)]$ and $L[t\sin(at)]$ shown below.

$$L[t\cos(at)] + \frac{a}{s}L[t\sin(at)] = \frac{1}{s}L[\cos(at)]$$

$$\frac{-a}{s}L[t\cos(at)] + L[t\sin(at)] = \frac{1}{s}L[\sin(at)]$$

Step 5: Substitute the known values of $L[\cos(at)]$ and $L[\sin(at)]$ into the system.

$$L[t\cos(at)] + \frac{a}{s}L[t\sin(at)] = \frac{1}{s}\left[\frac{s}{s^2+a^2}\right] = \frac{1}{s^2+a^2}$$
$$\frac{-a}{s}L[t\cos(at)] + L[t\sin(at)] = \frac{1}{s}\left[\frac{a}{s^2+a^2}\right] = \frac{a}{s(s^2+a^2)}$$

Step 6: Solve for $L[t\sin(at)]$ multiplying the first equation by $\dfrac{a}{s}$ and adding to the second, thus eliminating $L[t\cos(at)]$

$$\left(\frac{a}{s}\right)^2 L[t\sin(at)] + L[t\sin(at)] = \frac{a}{s(s^2+a^2)} + \frac{a}{s(s^2+a^2)} \Rightarrow$$
$$\left[\left(\frac{a}{s}\right)^2 + 1\right]L[t\sin(at)] = \frac{2a}{s(s^2+a^2)} \Rightarrow$$
$$\left[\frac{a^2+s^2}{s^2}\right]L[t\sin(at)] = \frac{2a}{s(s^2+a^2)} \Rightarrow L[t\sin(at)] = \frac{2as}{(a^2+s^2)^2} \therefore$$

Step 7: $L[t\cos(at)]$is obtained by simple substitution of the expression for $L[t\sin(at)]$into the first equation and solving for $L[t\cos(at)]$ via a moderate dose of algebra.

$$L[t\cos(at)] + \frac{a}{s}L[t\sin(at)] = \frac{1}{s^2+a^2} \Rightarrow$$
$$L[t\cos(at)] = \frac{1}{s^2+a^2} - \frac{a}{s}L[t\sin(at)] \Rightarrow$$
$$L[t\cos(at)] = \frac{1}{s^2+a^2} - \frac{2a^2}{(a^2+s^2)^2} \Rightarrow$$
$$L[t\cos(at)] = \frac{(s^2+a^2) - 2a^2}{(a^2+s^2)^2} \Rightarrow L[t\cos(at)] = \frac{s^2-a^2}{(a^2+s^2)^2} \therefore$$

The cyclic integration-by-parts technique is a great way to simultaneously generate antiderivatives for two closely related indefinite, definite, or improper integrals. *Note: I was first exposed to this in early 1967*. The same technique can be used to simultaneously generate $L[\cos(at)]$ and $L[\sin(at)]$, which I will leave as an exercise.