TRANSFORMS OF DERIVATIVES -

SOLUTIONS OF DIFFERENTIAL EQUATIONS

By using the definition of the Laplace transform and then integrating by parts we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

If f(t) is of order e^{at} as $t \to \infty$, then whenever s>a, the term $e^{-st} f(t)\Big]_0^\infty$ becomes –f(0), while $s\int_0^\infty e^{-st} f(t)dt$ becomes sF(s). So,

$$L\{f'(t)\} = sF(s) - f(0), \text{ where } F(s) = L\{f(t)\}.$$

We then have

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s[sF(s) - f(0)] - f'(0) \\ L\{f''(t)\} &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

To solve a differential equation we modify the notation slightly:

$$\begin{aligned} &L\{y(x)\} = Y(s);\ L^{-1}\{Y(s)\} = y(x), \text{ so} \\ &L\{y'\} = sY(s) - y(0) \\ &L\{y''\} = s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Note that to solve a differential equation using Laplace transforms we need to know the values of y(0) and y’(0). These are usually given and the problem is called an initial-value problem.

Example: Solve $y''(x) + y(x) = x$ given $y(0) = \pi$, and $y'(0) = 1$

Solution: First, we take the Laplace transform of each term of the equation:

$$L\{y''\} + L\{y\} = L\{x\}$$
$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2}$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + 1 + \pi s$$
$$(s^2+1)Y(s) = \frac{s^2+1}{s^2} + \pi s$$
$$Y(s) = \frac{1}{s^2} + \pi \frac{s}{s^2+1}$$

Now, taking inverse transforms we get

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + \pi L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

And the solution is: y(x) = x + π Cos x

Example: Solve $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$

Solution: Take the transform of each term of the equation.

$$L\{y''\} - L\{y'\} - 6L\{y\} = L\{2\}. \text{ Then}$$
$$s^2 Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = \frac{2}{s}$$
$$s^2 Y(s) - s - sY(s) + 1 - 6Y(s) = \frac{2}{s}$$

Solving for Y(s),

$$(s^2 - s - 6)Y(s) = \frac{s^2 - s + 2}{s}$$
$$Y(s) = \frac{s^2 - s + 2}{s(s-3)(s+2)}$$

Expanding the right side into partial fractions gives

$$Y(s) = -\frac{1}{3}\cdot\frac{1}{s} + \frac{8}{15}\cdot\frac{1}{s-3} + \frac{4}{5}\cdot\frac{1}{s+2}$$

Now, taking inverse transforms,

$$L^{-1}\{Y(s)\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{8}{15}L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{4}{5}L^{-1}\left\{\frac{1}{s+2}\right\}$$

We get the solution, $y(x) = -\frac{1}{3} + \frac{8e^{3x}}{15} + \frac{4e^{-2x}}{5}$.

Note: Here is a simplified method of expanding rational expressions into partial fractions. Any number which makes a factor of the denominator equal to zero is called a "pole". This method works provided the order (degree) of the factor is one. For example, if the denominator factors into s(s – 5)(s + 7), then there are three poles: 0, 5, and –7. If any of the factors were of degree higher than one, i.e. s^2 or $(s-5)^2$, this method would not work. We are going to use the expression from the previous example,

$\frac{s^2 - s + 2}{s(s-3)(s+2)}$. The poles are 0, 3, -2.

We have $\frac{s^2 - s + 2}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$

And we need to find A, B, and C.

For each pole, remove its factor and set

$$A = \lim_{s\to 0} \frac{s^2 - s + 2}{(s-3)(s+2)} = -\frac{1}{3}$$

$$B = \lim_{s\to 3} \frac{s^2 - s + 2}{s(s+2)} = \frac{8}{15}$$

$$C = \lim_{s\to -2} \frac{s^2 - s + 2}{s(s-3)} = \frac{4}{5}$$

So, $\dfrac{s^2 - s + 2}{s(s-3)(s+2)} = -\dfrac{1}{3s} + \dfrac{8}{15(s-3)} + \dfrac{4}{s(s+2)}$

<u>Example</u>: Solve $y'''(x) + y'(x) = e^{2x}, \quad y(0) = y'(0) = y''(0) = 0.$

Solution: For this problem we need to know that

$$L\{y'''\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$
$$L\{y'\} = sY(s) - y(0)$$
$$L\{e^{2x}\} = \frac{1}{s-2}$$

We then have

$$s^3 Y(s) + sY(s) = \frac{1}{s-2}$$
$$Y(s) = \frac{1}{s(s-2)(s^2+1)}$$

Since one of the factors in the denominator is not linear, we cannot use our simplified method for partial fractions unless we resort to complex numbers.

Setting $Y(s) = \dfrac{A}{s} + \dfrac{B}{s-2} + \dfrac{Cs+D}{s^2+1}$ and solving for A, B, C, and D, we get

$$Y(s) = -\frac{1}{2s} + \frac{1}{10(s-2)} + \frac{2s}{5(s^2+1)} - \frac{1}{5(s^2+1)}$$

Then, $$L^{-1}\{Y(s)\} = -\frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{10}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{5}L^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s^2+1}\right\},$$

And $$y(x) = -\frac{1}{2} + \frac{e^{2x}}{10} + \frac{2}{5}Cos\,x - \frac{1}{5}Sin\,x.$$

Exercises:

1. $y''(t) - y(t) = Sin\,t \quad y(0) = y'(0) = 0$
2. $y''(t) - 2y'(t) + y(t) = 0 \quad y(0) = 0,\ y(1) = 2$
3. $y''(t) - 2ky'(t) + (k^2 + h^2)y(t) = 0 \quad y(0) = 0,\ y'(0) = 1$
4. $y''(t) + 4y(t) = Sin\,t \quad y(0) = y'(0) = 0$
5. $y' + y = f(t) \quad y(0) = 0 \quad f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1 \end{cases}$

Answers:

1. $y = -1 + \frac{e^t}{2} + \frac{Cos\,t}{2} - \frac{Sin\,t}{2}$
2. $y(t) = 2te^{t-1}$
3. $y(t) = \frac{1}{h}e^{kt}Sinh\,t$
4. $y = \frac{Sinh\,2t}{10} - \frac{Sin\,t}{5}$
5. $y = 1 - e^{-t} - 2\left[1 - e^{-(t-1)}\right]u(t-1)$