THE UNIT STEP FUNCTIONS and LAPLACE TRANSFORMS

The Unit Step Function, u(t) is like an off-on switch that turns a function on or off over a specified interval. By definition,

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

If we consider u(t)f(t), then f(t) is turned off when $t < 0$ and turned on when $t \geq 0$. This is so because if $t < 0$, u(t)f(t)=(0)f(t)=0, and if $t \geq 0$, u(t)f(t)=(1)f(t)=f(t).

The translated unit step function, u(t-a) is defined as

$$u(t-a) = \begin{cases} 0 & 0 < t < a \\ 1 & t \geq a \end{cases}$$

So that $$u(t-2) = \begin{cases} 0 & 0 < t < 2 \\ 1 & t \geq 2 \end{cases}$$

Example: Suppose

$y = u(t-2)t^2$. Then for $t < 2$, $u(t - 2) = 0$ and $y = (0)t^2 = 0$. For $t \geq 2$, $u(t - 2) = 1$, so $y = (1)t^2 = t^2$.

Example: Suppose $f(t) = t^3$ only on the interval [2,3). Note that

$$u(t-2) = \begin{cases} 0 & 0 < t < 2 \\ 1 & t \geq 2 \end{cases} \qquad u(t-3) = \begin{cases} 0 & 0 < t < 3 \\ 1 & t \geq 3 \end{cases}$$

Now look at u(t-2) – u(t-3).

$$\text{If } \begin{array}{l} t < 2,\ u(t-2) - u(t-3) = 0 - 0 = 0 \\ 2 \leq t < 3\ u(t-2) - u(t-3) = 1 - 0 = 1 \\ t \geq 3\ u(t-2) - u(t-3) = 1 - 1 = 0 \end{array}$$

Then $[u(t-2) - u(t-3)]t^3$ will equal t^3 only on [2.3).

Suppose a function, f(t) is defined as follows:

$$f(t) = \begin{cases} 2 & 0 < t < 2 \\ -1 & 2 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$$

We wish to express f(t) in terms of unit step functions, u(t-2) and u(t-3). If t<2, both u(t-2) and u(t-3) are equal to zero. Set f(t)=2 as a start. If $2 \leq t < 3$, u(t-2)=1 and u(t-3)=0, and since f(t)=-1 set f(t)=2-3u(t-2). If $t \geq 3$, both u(t-2) and u(t-3) are equal to one, and since f(t)=1, set f(t)=2-3u(t-2) + 2u(t-3).

Suppose f(t) is defined as f(t)=$\begin{cases} t & 0 \le t < 2 \\ t_2 & t \ge 2 \end{cases}$. Express f(t) in terms of unit step functions. If $1 \le t < 2,\ u(t-1) = 1$ and $u(t-2) = 0.$ so set $f(t) = [u(t-1) - u(t-2)]t.$ If $t \ge 2,$ u(t-1)=1 and u(t-2)=1, so set $f(t) = [u(t-1) - u(t-2)]t + u(t-2)t^2.$

Note that the Laplace transform of the translated unit step function is

$$L\{u(t-a)\} = \frac{e^{-as}}{s}.$$

What is the Laplace transform of the function f(t) defined by

$$f(t) = \begin{cases} 2 & 0 < t < 2 \\ -1 & 2 \le t < 3 \\ 1 & t \ge 3 \end{cases} \quad ?$$

From above, f(t)=2-3u(t-2) + 2u(t-3).

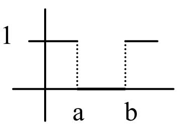
$$\begin{aligned} L\{f(t)\} &= L\{2\} - 3L\{u(t-2)\} + 2L\{u(t-3)\} \\ &= \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{2e^{-3s}}{s} \end{aligned}$$

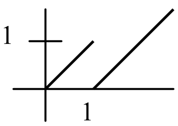
In general, if $f(t) = \begin{cases} g(t) & 0 \le t < a \\ h(t) & t \ge a \end{cases}$ then f(t)=g(t)-g(t)u(t-a)+h(t)u(t-a).

For the Laplace transform of the product of f(t-a) and the unit step function, u(t-a) see the worksheet on translation theorems.

Exercises: Write the following functions in terms of unit step functions. Then find the Laplace transform of the function.

1. $f(t)=\begin{cases}1 & 0<t<2\\ -1 & t\geq 2\end{cases}$ 2. $f(t)=\begin{cases}2 & 0\leq t<5\\ 0 & 5\leq t<7\\ -2 & t\geq 7\end{cases}$ 3. $f(t)=\begin{cases}t^2 & 0\leq t<1\\ t+2 & t\geq 1\end{cases}$

4. $f(t)=\begin{cases}0 & 0\leq t<\frac{3\pi}{2}\\ Cos\,t & t\geq\frac{3\pi}{2}\end{cases}$ 5. $f(t)=$ 

6. $f(t)=$ 

7. Find $L\{(t^2+5t)u(t-3)\}$.

Answers:

1. $f(t)=1-2u(t-2),\quad \frac{1}{s}\left(1-2e^{-2s}\right)$.

2. $f(t)=2-2u(t-5)-2u(t-7),\quad \frac{2}{s}\left(1-e^{-5s}-e^{-7s}\right)$.

3. $f(t)=t^2-u(t-1)t^2+u(t-1)(t+2),\quad \frac{2}{s^3}-\frac{2e^{-s}}{s^3}+\frac{3e^{-s}}{s^2}+\frac{4e^{-s}}{s}$.

4. $f(t)=u(t-\frac{3\pi}{2})Cos\,t,\quad \frac{e^{-\frac{3\pi}{2}}}{s^2+1}$.

5. $f(t)=1-u(t-a)+u(t-b),\quad \frac{1}{s}-\frac{e^{-as}}{s}+\frac{e^{-sb}}{s}$.

6. $f(t)=t-u(t-1)t+u(t-1)(t-1),\quad \frac{1}{s}-\frac{e^{-s}}{s}$.

7. $\frac{2e^{-3s}}{s^3}+\frac{11e^{-3s}}{s^2}+\frac{24e^{-3s}}{s}$.