Mechanics of Doing a Power Series Solution for an ODE

Before the advent of modern computing, many otherwise intractable ODEs were solved using power series having the general form $\sum_{i=0}^{\infty} a_i x^i$ or $\sum_{i=0}^{\infty} a_i (x-b)^i$. Once generated, power-series solutions allowed the computation of numerical answers to any degree of accuracy required as long as the technical practitioner was willing to put in the necessary dog work using a slide rule, log tables, etc. Needless to say, power-series solutions—*though still very much a part of the standard differential-equation curriculum*—have fallen by the wayside in the computer age. The example below illustrates the rather messy power-series technique for the ODE $y' = 2xy : y(0) = 3$ with closed-form solution $y(x) = 3e^{x^2}$.

Step 1: Let $y(x) = \sum_{i=0}^{\infty} a_i x^i$ which implies $y'(x) = \sum_{i=1}^{\infty} i a_i x^{i-1}$.

Note: Notice immediately how the indexing changes from $i = 0$ to $i = 1$. Keeping the indexing straight is critical to formulating a correct power-series solution! The form $y(x) = \sum_{i=0}^{\infty} a_i x^i$ is chosen in part because of the Initial condition $y(0) = 3$. An initial condition such as $y(2) = 3$ might be better served with a general series form such as $y(x) = \sum_{i=0}^{\infty} a_i (x-2)^i$, which would allow for an easier determination of the unknown coefficients.

Step 2: Apply the Initial Condition $y(0) = 3$ which implies $a_0 = 3$.

Step 3: Substitute the expressions for $y(x)$ and $y'(x)$ into $y' = 2xy$

$$\sum_{i=1}^{\infty} i a_i x^{i-1} = 2x \sum_{i=0}^{\infty} a_i x^i \Rightarrow \sum_{i=1}^{\infty} i a_i x^{i-1} = 2 \sum_{i=0}^{\infty} a_i x^{i+1} \Rightarrow$$

$$\sum_{i=0}^{\infty} (i+1) a_{i+1} x^i = 2 \sum_{i=0}^{\infty} a_i x^{i+1} \Rightarrow$$

$$\sum_{i=0}^{\infty} \left[(i+1) a_{i+1} x^i - 2 a_i x^{i+1} \right] = 0 \Rightarrow$$

$$\{a_1 - 2a_0 x\} + \{2a_2 x - 2a_1 x^2\} + \{3a_3 x^2 - 2a_2 x^3\} + \{4a_4 x^3 - 2a_3 x^4\} +$$
$$\{5a_5 x^4 - 2a_4 x^5\} + \{6a_6 x^5 - 2a_5 x^6\} + \{7a_7 x^6 - 2a_6 x^7\} + \ldots = 0$$

Note: Keep your indexes straight and press!

Step 4: Gather coefficients for like powers of x into like-power groups.

$$\{a_1\} x^0 + \{2a_2 - 2a_0\} x^1 + \{3a_3 - 2a_1\} x^2 + \{4a_4 - 2a_2\} x^3 + \{5a_5 - 2a_3\} x^4 +$$
$$\{6a_6 - 2a_4\} x^5 + \{7a_7 - 2a_5\} x^6 + \{8a_8 - 2a_6\} x^7 + \ldots = 0$$

Note: Critical here is being able to recognize and continue a general pattern.

Step 5: Set each coefficient group equal to zero—why!—which leads to two iterative cascades.

$$1: a_1 = 0 \Rightarrow a_3 = a_5 = a_7 = a_9 = ... = 0$$
$$2: a_0 = 3 \Rightarrow a_2 = a_0 \Rightarrow a_4 = a_2 / 2 \Rightarrow a_6 = a_4 / 3 \Rightarrow a_8 = a_6 / 4 \Rightarrow a_{10} = a_8 / 5 \Rightarrow ...$$

Step 6: Write the long series form of the solution (without indices) carefully matching coefficients to their original powers as found in $y(x) = \sum_{i=0}^{\infty} a_i x^i$

$$y(x) = 3 + 3x^2 + \frac{3}{2}x^4 + \frac{3}{2\cdot 3}x^6 + \frac{3}{2\cdot 3\cdot 4}x^8 + \frac{3}{2\cdot 3\cdot 4\cdot 5}x^{10} + ...$$

Step 7: Rewrite in condensed form—watch this and watch the indexing!

$$y(x) = 3 + 3x^2 + \frac{3}{2}x^4 + \frac{3}{2\cdot 3}x^6 + \frac{3}{2\cdot 3\cdot 4}x^8 + \frac{3}{2\cdot 3\cdot 4\cdot 5}x^{10} + ... =$$
$$3\left\{1 + x^2 + \frac{1}{2}x^4 + \frac{1}{2\cdot 3}x^6 + \frac{1}{2\cdot 3\cdot 4}x^8 + \frac{1}{2\cdot 3\cdot 4\cdot 5}x^{10} + ...\right\} =$$
$$3\left\{1 + (x^2)^1 + \frac{1}{2}(x^2)^2 + \frac{1}{2\cdot 3}(x^2)^3 + \frac{1}{2\cdot 3\cdot 4}(x^2)^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5}(x^2)^5 + ...\right\} = 3\sum_{i=0}^{\infty}\frac{(x^2)^i}{i!}$$

The student should recognize the last expression $3\sum_{i=0}^{\infty}\frac{(x^2)^i}{i!}$ as the power series for $3e^{x^2}$, obtained from the power series for $e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}$ by simply substituting x^2 for x and multiplying by 3. Thus the solution for this particular ODE, obtained by the power-series method, matches the solution obtained via a classical separation-of-variables method.

Note: Some fairly heavy mathematical issues surround the use of power-series methods such as when does the series solution converge, how many terms are needed for a particular degree of accuracy, etc. When I first started working at the WPAFB, series solutions were still being used for simple first cuts at engineering problems (my area of concentration was heat transfer). We would program a series solution on a computer and let the computer tell us if it converged or not simply by examining the output. Stable non-oscillating numerical output trending right and not 'bombing' the machine was ***assumed*** *to be good* for preliminary calculations assessing trends. *A subsequent test program was the proof.*

Homework Assignment

Page 279: Problems 13 and 14 with $y(0) = 0, y'(0) = 1$.

Also, work the ODE $y' = 3x^2 y : y(0) = 1$.