Notes on The Laplace Transform

Definition (7.1)

The Laplace transform is an integral transform which, if it converges, is defined as:

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

The transform of a linear combination of functions is a linear combination of transforms, that is:

$$\mathscr{L}\{f(t) + g(t)\} = F(s) + G(s)$$

so the Laplace transform may exist for continuous piecewise functions.

<u>Basic functions</u>

$$\mathscr{L}\{1\} = \frac{1}{s} \qquad \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2} \qquad \mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \qquad \mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

Inverse Transforms (7.2.1)

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}^{-1}\{F(s)\} = f(t)$. The inverse Laplace is also a linear transform.

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \qquad \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \qquad \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt \qquad \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$$

Transforms of Derivatives (7.2.2)

$$\mathscr{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

For example: $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$

$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

Transforms and inverse transforms can be used to solve a differential equation by first using transforms to create an algebraic equation in F(s) and then using the inverse transforms to solve for f(t).

Examples

1) Solve: $y' + 6y = 3e^{4t}$ $y(0) = 2$

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 6\mathscr{L}\{y\} = 3\mathscr{L}\left\{e^{4t}\right\}$$

$$sY(s) - y(0) + 6Y(s) = \frac{3}{s-4}$$

$$Y(s)[s+6] - 2 = \frac{3}{s-4}$$

$$Y(s) = \frac{3}{(s-4)(s+6)} + \frac{2}{s+6} = \frac{3}{10}\left(\frac{1}{s-4}\right) + \frac{17}{10}\left(\frac{1}{s+6}\right)$$

$$\mathscr{L}^{-1}\left\{Y(s)\right\} = \frac{3}{10}\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\} + \frac{17}{10}\mathscr{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$y(t) = \frac{3}{10}e^{4t} + \frac{17}{10}e^{-6t}$$

2) Solve: $y'' + 16y = e^{t}$ $y(0) = 0$ $y'(0) = 0$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} + 16\mathscr{L}\{y\} = \mathscr{L}\left\{e^{t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{1}{s-1}$$

$$Y(s)[s^2+16] = \frac{1}{s-1}$$

$$Y(s) = \frac{1}{(s-1)(s^2+16)} = \frac{1}{17}\left(\frac{1}{s-1}\right) - \frac{1}{17}\left(\frac{s}{s^2+16}\right) - \frac{1}{17}\left(\frac{1}{s^2+16}\right)$$

$$\mathscr{L}^{-1}\left\{Y(s)\right\} = \frac{1}{17}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{17}\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} - \frac{1}{17}\mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$$

$$y(t) = \frac{1}{17}e^{t} - \frac{1}{17}\cos 4t - \frac{1}{68}\sin 4t$$

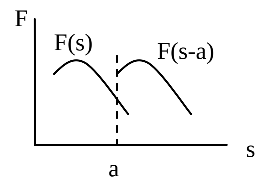
Operational Properties I (7.3)

Translation on the s-Axis (7.3.1)

If we know the Laplace transform of a function, then the transform of an exponential multiple of that function is a horizontal shift of the original transform on the s-axis.

That is: If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$

and: $\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$



Examples

1) $\mathscr{L}\{e^{5t}t\}$: $\mathscr{L}\{t\} = \dfrac{1}{s^2} \rightarrow \mathscr{L}\{e^{5t}t\} = \dfrac{1}{(s-5)^2}$

2) $\mathscr{L}\{e^{-2t}\sin 4t\}$: $\mathscr{L}\{\sin 4t\} = \dfrac{4}{s^2+16} \rightarrow \mathscr{L}\{e^{-2t}\sin 4t\} = \dfrac{4}{(s+2)^2+16}$

3) $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)^3}\right\}$: $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\dfrac{2}{s^3}\right\} = \dfrac{1}{2}t^2 \rightarrow \mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)^3}\right\} = \dfrac{1}{2}e^{-t}t^2$

4) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+2s+7}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)^2+6}\right\}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+6}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s^2+6)}\right\} = \frac{1}{\sqrt{6}}\sin\sqrt{6}\,t$$

$$\rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+6}\right\} = \frac{1}{\sqrt{6}}e^{-t}\sin\sqrt{6}\,t$$

5) Solve: $y'' - 2y' + y = t^3e^t \qquad y(0) = 0 \qquad y'(0) = 0$

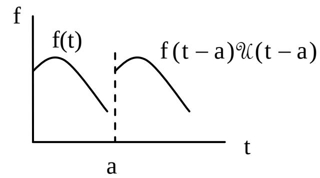
$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = \frac{3!}{(s-1)^4}$$

$$Y(s)[s^2 - 2s + 1] = \frac{3!}{(s-1)^4}$$

$$Y(s) = \frac{3!}{(s-1)^6} = \frac{1}{20}\frac{5!}{(s-1)^6} \qquad \rightarrow \qquad y(t) = \frac{1}{20}t^5e^t$$

Translation on the t-Axis (7.3.2)

If a transform F(s) is multiplied by e^{-as}, $a > 0$, the inverse transform is the function f(t) translated along the t-axis and involves the unit step function $\mathscr{U}(t-a)$.



$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s) \qquad \text{where } F(s) = \mathscr{L}\{f(t)\}$$

Alternative form: $\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

The unit step function is: $\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

and the piecewise function $f(t-a)\mathscr{U}(t-a)$ is: $f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$

The piecewise function: $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$

can be written: $f(t) = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$

The piecewise function: $f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$

can be written: $f(t) = g(t)[\,\mathscr{U}(t-a) - \mathscr{U}(t-b)\,]$

Examples

1) Rewrite using $\mathscr{U}(t-a)$: $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$ $\rightarrow$ $t^2\mathscr{U}(t-2)$

2) Rewrite using $\mathscr{U}(t-a)$: $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$ $\rightarrow$ $t - t\mathscr{U}(t-1)$

3) Rewrite using $\mathscr{U}(t-a)$: $f(t) = \begin{cases} t, & 0 \le t < 3 \\ 0, & 3 \le t < 4 \\ t^2, & t \ge 4 \end{cases}$ $\rightarrow$ $t - t\mathscr{U}(t-3) + t^2\mathscr{U}(t-4)$

3) Find $\mathscr{L}\{t\mathscr{U}(t-3)\}$

a) Rewrite as: $\mathscr{L}\{(t-3+3)\mathscr{U}(t-3)\} = \mathscr{L}\{(t-3)\mathscr{U}(t-3)\} + \mathscr{L}\{3\mathscr{U}(t-3)\}$

$$= e^{-3s}\frac{1}{s^2} + e^{-3s}\frac{3}{s} = e^{-3s}\left[\frac{1}{s^2}+\frac{3}{s}\right]$$

b) Alternative form: $\mathscr{L}\{t\mathscr{U}(t-3)\} = e^{-3s}\mathscr{L}\{t+3\} = e^{-3s}\left[\frac{1}{s^2}+\frac{3}{s}\right]$

4) Find $\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s^4}\right\}$

$$\mathscr{L}^{-1}\left\{e^{-3s}\frac{1}{s^4}\right\} = f(t-3)\,\mathscr{U}(t-3)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = f(t) = \frac{1}{6}t^3 \quad \rightarrow \quad f(t-3) = \frac{1}{6}(t-3)^3$$

$$\mathscr{L}^{-1}\left\{e^{-3s}\frac{1}{s^4}\right\} = \frac{1}{6}(t-3)^3\,\mathscr{U}(t-3)$$

5) Solve: $y' + y = f(t)$ $\quad y(0) = 0$ $\quad f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 3, & t \ge 2 \end{cases}$

$y' + y = 3\mathscr{U}(t-2)$ $\qquad \mathscr{L}\{3\mathscr{U}(t-2)\} = 3e^{-2s}\left(\frac{1}{s}\right)$

$$sY(s) - y(0) + Y(s) = 3e^{-2s}\left(\frac{1}{s}\right)$$

$$Y(s)[s+1] = 3e^{-2s}\left(\frac{1}{s}\right)$$

$$Y(s) = 3e^{-2s}\left(\frac{1}{s(s+1)}\right) = 3e^{-2s}\left(\frac{1}{s}\right) - 3e^{-2s}\left(\frac{1}{s+1}\right)$$

$f(t) = 1$ $\qquad f(t) = e^{-t}$

$f(t-2) = e^{-(t-2)}$

$$y(t) = 3\,\mathscr{U}(t-2) - 3e^{-(t-2)}\mathscr{U}(t-2)$$

6) Solve: $y'' + 4y = (\cos t)\mathscr{U}(t-2\pi)$ $\qquad y(0) = 1$ $\qquad y'(0) = 0$

$$\mathscr{L}\{(\cos t)\mathscr{U}(t-2\pi)\} = e^{-2\pi s}\mathscr{L}\{\cos(t+2\pi)\} = e^{-2\pi s}\mathscr{L}\{\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = e^{-2\pi s}\frac{s}{s^2+1}$$

$$Y(s)[s^2+4] = s + e^{-2\pi s}\frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+4} + e^{-2\pi s}\frac{s}{(s^2+1)(s^2+4)} = \frac{s}{s^2+4} + \frac{e^{-2\pi s}}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$y(t) = \cos 2t + \frac{1}{3}\left[\cos(t-2\pi) - \cos(2t-2\pi)\right]\mathcal{U}(t-2\pi)$$

$$y(t) = \cos 2t + \frac{1}{3}\left[\cos t - \cos 2t\right]\mathcal{U}(t-2\pi)$$

Operational Properties II (7.4)

Derivatives of a Transform (7.4.1)

If a function f(t) is multiplied by a power of t, the transform of the product will be a derivative of the transform of f(t):

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s) \qquad \text{where } F(s) = \mathscr{L}\{f(t)\} \text{ and } n = 1, 2, 3,\ldots$$

Examples

1) $\mathscr{L}\{t\sin 2t\} = (-1)\dfrac{d}{ds}\left[\dfrac{2}{s^2+4}\right] = \dfrac{4s}{(s^2+4)^2}$

2) $\mathscr{L}\{te^{2t}\sin 3t\} = (-1)\dfrac{d}{ds}\left[\dfrac{3}{(s-2)^2+9}\right] = \dfrac{6(s-2)}{\left[(s-2)^2+9\right]^2}$

3) $\mathscr{L}\{t^2\cosh t\} = (-1)^2\dfrac{d^2}{ds^2}\left[\dfrac{s}{s^2-1}\right] = \dfrac{2s(s^2+3)}{(s^2-1)^3}$

4) Solve: $y' - y = te^t\sin 2t \qquad y(0) = 0$

$$sY(s) - y(0) - Y(s) = (-1)\frac{d}{ds}\left[\frac{2}{(s-1)^2+4}\right]$$

$$Y(s)[s-1] = \frac{4(s-1)}{\left[(s-1)^2+4\right]^2} \quad\rightarrow\quad Y(s) = \frac{4}{\left[(s-1)^2+4\right]^2} = \frac{1}{4}\frac{16}{\left[(s-1)^2+4\right]^2}$$

$y(t) = \dfrac{1}{4}e^t(\sin 2t - 2t\cos 2t)$ (from table, with translation on the s-axis)

Transforms of Integrals (7.4.2)

If functions f and g are piecewise continuous on [0, ∞), the convolution f * g is:

$$f * g = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

The transform of the convolution is:

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

and: $$\mathscr{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

If $g(t) = 1$, then $G(s) = \frac{1}{s}$, and: $\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$ and $\mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$

<u>Examples</u>

1) $\mathscr{L}\left\{e^t * e^t \cos t\right\} = F(s)G(s) = \left(\frac{1}{s-1}\right)\left(\frac{s-1}{(s-1)^2+1}\right) = \left(\frac{1}{(s-1)^2+1}\right)$

2) $\mathscr{L}\left\{\int_0^t \tau\, e^{(t-\tau)}\,d\tau\right\} = \mathscr{L}\{t\}\mathscr{L}\left\{e^t\right\} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$

3) $\mathscr{L}\left\{\int_0^t \tau\, e^{\tau}\,d\tau\right\}$ $\quad g(t) = 1 \quad G(s) = \frac{1}{s}$

$f(t) = te^t \quad F(s) = \mathscr{L}\left\{te^t\right\} = (-1)\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$

$$F(s)G(s) = \frac{1}{s}\frac{1}{(s-1)^2}$$

4) $\mathscr{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$ $\quad G(s) = \frac{1}{s} \quad g(t) = 1 \quad F(s) = \frac{1}{s+1} \quad f(t) = e^{-t}$

$$= \int_0^t e^{-\tau}\,d\tau = -e^{-t} + 1$$

5) $\mathscr{L}^{-1}\left\{\frac{1}{s(s-2)^2}\right\}$ $\quad G(s) = \frac{1}{s} \quad g(t) = 1 \quad F(s) = \frac{1}{(s-2)^2} \quad f(t) = te^{2t}$

$$= \int_0^t \tau e^{2\tau}\,d\tau = \frac{t}{2}e^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}$$

6) Solve: $y' + 4y + 4\int_0^t y(\tau)\,d\tau = 1 \qquad y(0) = 0$

$$sY(s) - y(0) + 4Y(s) + 4\frac{Y(s)}{s} = \frac{1}{s}$$

$$Y(s)\left[s + 4 + \frac{4}{s}\right] = \frac{1}{s}$$

$$Y(s) = \frac{1}{(s+2)^2}$$

$$y(t) = te^{-2t}$$

Transform of a Periodic Function (7.4.3)

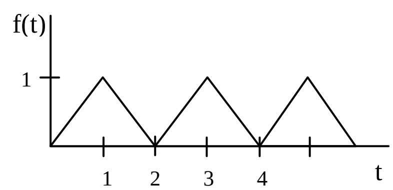
If f(t) is piecewise continuous on [0, ∞), of exponential order, and periodic with period T, then:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Example

Find the transform of the given periodic function:

For $0 \le t < 2$ and $T = 2$:



$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\left[\int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}(s+1)}{s^2} + \frac{e^{-s}(s-1)+e^{-2s}}{s^2}\right]$$

$$= \frac{(e^{-s}-1)^2}{s^2(1-e^{-2s})}$$

$$= \frac{(e^{-s}-1)^2}{s^2(1-e^{-s})(1+e^{-s})}$$

$$= \frac{1-e^{-s}}{s^2(1+e^{-s})}$$

$$= \frac{e^s-1}{s^2(e^s+1)}$$