Notes on Linear Models: Initial-Value Problems

Spring/Mass Systems: Free Undamped Motion (5.1.1)

For a mass *m* suspended from a spring, the weight is jointly proportional to the distance stretched and a constant: F = ks (Hooke's Law) or weight = kx.

If the motion of the mass is free and undamped (also called simple harmonic), it is described by:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{circular frequency:} \quad \omega = \sqrt{\frac{k}{m}} \quad \text{(in radians per second)}$$

$$\text{period:} \quad T = \frac{2\pi}{\omega} \qquad \text{frequency:} \quad f = \frac{1}{T}$$

The equation of motion is: $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$

$x(0) = x_0$ = initial displacement: positive if *below* equilibrium point
negative if *above* equilibrium point

$x'(0) = x_0'$ = initial velocity: positive *downward*
negative *upward*
= 0 if from rest

Units of measurement: $\text{mass} = \dfrac{\text{weight}}{g}$ $\qquad \text{slugs} = \dfrac{\text{lbs}}{32\,\text{ft}/\text{s}^2}$ $\qquad \text{kg} = \dfrac{\text{N}}{9.8\,\text{m}/\text{s}^2}$

<u>Example</u>

A mass weighing 24 lbs stretches a spring 3 inches. The mass is released from rest 4 inches above the equilibrium position. Find the equation of motion.

a) $x_0 = -1/3$ ft $\qquad x_0' = 0$

b) $k = \dfrac{W}{x} = \dfrac{24 \text{ lb}}{1/4\,\text{ft}} = 96$ ft/lb

c) $m = \dfrac{24 \text{ lb}}{32} = \dfrac{3}{4}$ slugs

d) $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{96}{3/4}} = 8\sqrt{2}$

e) $x(t) = c_1 \cos 8\sqrt{2}\,t + c_2 \sin 8\sqrt{2}\,t$

$x(0) = -1/3 = c_1 \quad \rightarrow \quad x(t) = -\dfrac{1}{3}\cos 8\sqrt{2}\,t + c_2 \sin 8\sqrt{2}\,t$

$x'(t) = \dfrac{8\sqrt{2}}{3}\sin 8\sqrt{2}\,t + 8\sqrt{2}\,c_2 \cos 8\sqrt{2}\,t$

$x'(0) = 0 \quad \rightarrow \quad c_2 = 0$

f) equation of motion: $x(t) = -\dfrac{1}{3}\cos 8\sqrt{2}\,t$

Alternative form of equation of motion:

$$x(t) = A\sin(\omega t+\varphi) \qquad A = \sqrt{c_1^{\,2}+c_2^{\,2}} \qquad \tan\varphi = \frac{c_1}{c_2}$$

A is the amplitude of the vibrations; φ is the phase angle (in radians).

The quadrant of φ is determined by looking at: $\sin\varphi = \dfrac{c_1}{A}$ and $\cos\varphi = \dfrac{c_2}{A}$

Example

A mass weighing 6.4 lb stretches a spring 1/4 ft. It is removed and a mass of 1.6 slugs is attached and released from an initial position of 1/3 ft above equilibrium with a downward velocity of 1 ft/s. Find the alternative form of the equation of motion. When does the mass pass through the equilibrium point? When is it first heading upward with a velocity of 1.5 ft/s?

a) $x_0 = -1/3 \qquad x_0' = 1$

b) $k = \dfrac{W}{x} = \dfrac{6.4\text{ lb}}{1/4\,\text{ft}} = 25.6$ ft/lb

c) $m = 1.6$ slugs

d) $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{25.6}{1.6}} = 4$

e) $x(t) = c_1\cos 4t + c_2\sin 4t$

$$x(0) = -\frac{1}{3} = c_1 \quad\rightarrow\quad x(t) = -\frac{1}{3}\cos 4t + c_2\sin 4t$$

$$x'(t) = \frac{4}{3}\sin 4t + 4c_2\cos 4t$$

$$x'(0) = 1 = 4c_2 \quad\rightarrow\quad c_2 = \frac{1}{4}$$

equation of motion: $x(t) = -\dfrac{1}{3}\cos 4t + \dfrac{1}{4}\sin 4t$

f) $A = \sqrt{\left(-\dfrac{1}{3}\right)^2+\left(\dfrac{1}{4}\right)^2} = \dfrac{5}{12} \qquad \tan\varphi = \dfrac{-1/3}{1/4} \qquad \varphi$ is in QIV; $\varphi = -0.92730$

g) alt. equation of motion: $x(t) = \dfrac{5}{12}\sin(4t-0.92730) \qquad$ period $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

h) equilibrium position occurs when $x(t) = 0$, or at $4t - 0.92730 = 0$, or $t = 0.23183 + \dfrac{n\pi}{4}$

i) $x'(t) = -1.5 = \dfrac{5}{3}\cos(4t-0.92730) \quad\rightarrow\quad t = 0.90447$ s

(this velocity will occur at $0.90447 + \dfrac{n\pi}{2}$)

Spring/Mass Systems: Free Damped Motion (5.1.2)

The differential equation for free damped motion is:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \qquad \text{where } \omega^2 = \frac{k}{m}, \quad 2\lambda = \frac{\beta}{m}, \text{ and } \beta = \text{damping constant}$$

The auxiliary equation $m^2 + 2\lambda m + \omega^2 = 0$ has roots m_1, m_2 of $-\lambda \pm \sqrt{\lambda^2 - \omega^2}$.

If $\lambda^2 - \omega^2 > 0$, then system is *overdamped* (large β relative to k); motion is non-oscillatory.

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

If $\lambda^2 - \omega^2 = 0$, the system is *critically damped*; any decrease in β would result in oscillations.

$$x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t} = e^{-\lambda t}(c_1 + c_2 t)$$

If $\lambda^2 - \omega^2 < 0$, the system is *underdamped* (small β relative to k); motion is oscillatory, but the amplitudes → 0 as t → ∞.

$$\text{roots} = -\lambda \pm \sqrt{\omega^2 - \lambda^2}\; i$$

$$x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t\right)$$

<u>Example</u>

A mass weighing 16 lbs is attached to a spring with a constant of 2 lb/ft. The damping force is numerically twice the instantaneous velocity (β = 2). The mass is released from 1 ft above the equilibrium position with a downward velocity of 8 ft/s. When does the mass go through the equilibrium position? What is its extreme displacement?

a) $x_0 = -1 \qquad x_0' = 8$

b) $k = 2 \qquad \beta = 2 \qquad m = \frac{16\,\text{lb}}{32} = \frac{1}{2}\text{ slug}$

c) $\omega^2 = \frac{k}{m} = \frac{2}{1/2} = 4 \qquad 2\lambda = \frac{\beta}{m} = \frac{2}{1/2} = 4 \qquad \lambda = 2$

d) $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0 \qquad \lambda^2 - \omega^2 = 4 - 4 = 0$ critically damped

e) $m^2 + 4m + 4 = 0 \quad \rightarrow \quad m = 2, 2$

f) $x(t) = c_1 e^{-2t} + c_2 t e^{-2t} = e^{-2t}(c_1 + c_2 t)$

g) $x(0) = -1 = c_1$

$x(t) = -e^{-2t} + c_2 t e^{-2t}$

$x'(t) = 2e^{-2t} + c_2e^{-2t} - 2c_2te^{-2t}$

$x'(0) = 8 = 2 + c_2 \quad \rightarrow \quad c_2 = 6$

equation of motion: $x(t) = -e^{-2t} + 6te^{-2t}$

h) equilibrium position: $x = 0$

$0 = -e^{-2t}(1 - 6t)$

$1 - 6t = 0$

$t = 1/6$ s

i) $x'(t) = 8e^{-2t} - 12te^{-2t}$

$0 = 8 - 12t \quad \rightarrow$

$t = 2/3$ s

$x(2/3) = 3e^{(-4/3)} \approx 0.791$ ft

The alternative form for an underdamped system is:

$$x(t) = Ae^{-\lambda t}\sin\left(\sqrt{\omega^2 - \lambda^2}\,t + \varphi\right) \qquad A = \sqrt{c_1^{\,2} + c_2^{\,2}} \qquad \tan\varphi = \frac{c_1}{c_2}$$

$$\sin\varphi = \frac{c_1}{A} \qquad \cos\varphi = \frac{c_2}{A}$$

$Ae^{-\lambda t}$ is the damped amplitude. The time period between two successive maxima of x(t) is the quasi period $\dfrac{2\pi}{\sqrt{\omega^2 - \lambda^2}}$ and $\dfrac{\sqrt{\omega^2 - \lambda^2}}{2\pi}$ is the quasi frequency.

<u>Example</u>

A mass weighing 16 lbs is attached to a spring with a constant of 5 lb/ft. The damping force is numerically equal to the instantaneous velocity ($\beta = 1$). The mass is released from 1/2 ft below the equilibrium position with a downward velocity of 1 ft/s. Find the alternative form of the equation of motion. When does the mass first go through the equilibrium position heading upward?

a) $x_0 = ½ \qquad x'_0 = 1$

b) $k = 5 \qquad \beta = 1 \qquad m = \dfrac{16\,\text{lb}}{32} = \dfrac{1}{2}\,\text{slug}$

c) $\omega^2 = \dfrac{k}{m} = \dfrac{5}{1/2} = 10 \qquad 2\lambda = \dfrac{\beta}{m} = \dfrac{1}{1/2} = 2 \qquad \lambda = 1$

d) $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 10x = 0 \qquad \lambda^2 - \omega^2 = 1 - 10 = -9 < 0$ underdamped

e) $x(t) = e^{-t}(c_1\cos 3t + c_2\sin 3t)$

f) $x(0) = ½ = c_1$

$$x'(t) = -\frac{1}{2}e^{-t}\cos 3t - \frac{3}{2}e^{-t}\sin 3t - c_2 e^{-t}\sin 3t + 3c_2 e^{-t}\cos 3t$$

$x'(0) = 1 = -½ + 3c_2 \quad \rightarrow \quad c_2 = ½$

equation of motion: $x(t) = ½\, e^{-t}(\cos 3t + \sin 3t)$

g) $A = \sqrt{c_1^{\,2} + c_2^{\,2}} = \dfrac{\sqrt{2}}{2}$ $\qquad \tan\varphi = \dfrac{c_1}{c_2} = 1 \qquad c_1, c_2 > 0 \quad \rightarrow \varphi = \pi/4$

h) alt. equation of motion: $x(t) = \dfrac{\sqrt{2}}{2}e^{-t}\sin\left(3t + \dfrac{\pi}{4}\right)$

i) $x(t) = 0$ when $3t + \pi/4 = n\pi$, or $t = \dfrac{4n\pi - \pi}{12}$

when $n = 1$, $t = \pi/4 \approx .785$ s

$$x'(t) = -\frac{\sqrt{2}}{2}e^{-t}\sin\left(3t + \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}e^{-t}\cos\left(3t + \frac{\pi}{4}\right)$$

upward means $x' < 0$: when $t = \pi/4$, $3t + \pi/4 = \pi$, and the cos is negative.

So, the mass goes through equilibirium heading upward at $t = \pi/4$.

Spring/Mass Systems: Driven Motion (5.1.3)

The differential equation for driven or forced motion *with damping* is:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \qquad \text{where } F(t) = f(t)/m \text{ and } f(t) = \text{an external force}$$

$$\omega^2 = \frac{k}{m}, \quad 2\lambda = \frac{\beta}{m}, \text{ and } \beta = \text{damping constant}$$

The differential equation for driven or forced motion *without damping* is:

$$\frac{d^2x}{dt^2} + \omega^2 x = F(t) \qquad \text{since } \beta = 0$$

F(t) may be a periodic function, like $F_0 \sin\gamma t$ or $F_0 \cos\gamma t$. In this case, the complementary solution to a damped system, which decreases over time because of the $e^{-\lambda t}$ term, it is called the transient term or transient solution. The particular solution is the steady-state term or solution.

For example, if the periodic force is $F_0 \sin\gamma t$ with no damping:

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\gamma t \qquad \text{where } F_0 \text{ is a constant and } \gamma \neq \omega$$

The general solution is:

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t + \frac{F_0}{\omega^2 - \gamma^2}\sin\gamma t$$

As the frequency of the driving force $(\gamma/2\pi)$ approaches the frequency of free vibrations $(\omega/2\pi)$, the vibrations will increase in amplitude until the system fails.