Notes on Higher Order Differential Equations

Reduction of Order (by formula)   (4.2)

The general solution of a homogeneous linear second-order differential equation
$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ is the linear combination $y = c_1y_1 + c_2y_2$.

If y_1 is known, y_2 can be determined by substituting $y_2(x) = u(x)y_1(x)$ into the equation and solving for $u(x)$, which involves solving a first-order equation after substitutions.

For the standard form $y'' + P(x)y' + Q(x)y = 0$, the formula for y_2 is: $\displaystyle y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx$

Any coefficient that results in the integration can be discarded, since it is considered to be part of c_2.

<u>Examples</u>

1) $y'' - 2y' + y = 0$ $y_1 = xe^x$ $P(x) = -2$

$$y_2 = xe^x\int \frac{e^{-\int -2dx}}{(xe^x)^2}\,dx = xe^x\int \frac{e^{2x}}{x^2e^{2x}}\,dx = xe^x\int x^{-2}\,dx = xe^x(-x^{-1}) = -e^x$$

$y_2 = e^x$ (discard the coefficient)

general solution: $y = c_1xe^x + c_2e^x$

2) $4y'' + y' - (3/2)y = 0$ → $y'' + (1/4)y' - (3/8)y = 0$

$y_1 = e^{x/2}$ $P(x) = ¼$

$$y_2 = e^{x/2}\int \frac{e^{-\int (1/4)dx}}{(e^{x/2})^2}\,dx = e^{x/2}\int \frac{e^{-x/4}}{e^x}\,dx = e^{x/2}\int e^{-5x/4}\,dx = e^{x/2}\left(-\frac{4}{5}e^{-5x/4}\right) = -\frac{4}{5}e^{-3x/4}$$

$y_2 = e^{-3x/4}$

general solution: $y = c_1e^{x/2} + c_2e^{-3/4}$

3) $x^2y'' - 3xy' + 5y = 0$ → $y'' - (3/x)y' + (5/x^2)y = 0$

$y_1 = x^2\sin(\ln x)$ $P(x) = -3/x$

$$y_2 = x^2\sin(\ln x)\int \frac{e^{-\int(-3/x)dx}}{\left[x^2\sin(\ln x)\right]^2}\,dx = x^2\sin(\ln x)\int \frac{e^{3\ln x}}{x^4\sin^2(\ln x)}\,dx$$

$$= x^2\sin(\ln x)\int \frac{x^3}{x^4\sin^2(\ln x)}dx = x^2\sin(\ln x)\int \frac{1}{x}\csc^2(\ln x)\,dx = x^2\sin(\ln x)\left[-\cot(\ln x)\right]$$

$$= -x^2\cos(\ln x)$$

$y_2 = x^2\cos(\ln x)$ general solution: $y = c_1x^2\sin(\ln x) + c_2x^2\cos(\ln x)$

Homogeneous Linear Equations with Constant Coefficients (4.3)

For homogeneous linear equations, assume $y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2e^{mx}$, etc.
Since $e^{mx} \neq 0$, then the coefficients must add to equal 0.

For: $ay'' + by' + cy = 0 \rightarrow am^2e^{mx} + bme^{mx} + ce^{mx} = 0$
or: $am^2 + bm + c = 0$ ← the **auxiliary equation** of the differential equation

and: $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For two different real roots m_1 and m_2:	$y = c_1e^{m_1x} + c_2e^{m_2x}$
For two real repeated roots:	$y = c_1e^{mx} + c_2xe^{mx}$
For two complex roots $\alpha \pm \beta i$	$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$

For higher order equations, find the roots and apply the basic forms. Repeated roots require successive multiplication by x, e.g., $c_1e^{mx} + c_2xe^{mx} + c_3x^2e^{mx} + \ldots$

Examples

1) $y'' + 2y' - 8y = 0$
$m^2 + 2m - 8 = 0$
$(m + 4)(m - 2) = 0$ → $y = c_1e^{-4x} + c_2e^{2x}$

2) $y'' + 6y' + 9y = 0$
$m^2 + 6m + 9 = 0$
$(m + 3)^2 = 0$ → $y = c_1e^{-3x} + c_2xe^{-3x}$

3) $y'' - 2y' + 5y = 0$
$m^2 - 2m + 5 = 0$
$m = 1 \pm 2i$ → $y = c_1e^{x}\cos 2x + c_2e^{x}\sin 2x$

4) $y^{(4)} - 8y'' + 16y = 0$
$m^{(4)} - 8m^2 + 16 = 0$
$(m^2 - 4)^2 = 0$
$(m + 2)^2\,(m - 2)^2 = 0$ → $y = c_1e^{-2x} + c_2xe^{-2x} + c_3e^{2x} + c_4xe^{2x}$

5) $y''' - 6y'' + 12y' - 8y = 0$
$m^3 - 6m^2 + 12m - 8 = 0$
$(m - 2)^3 = 0$ → $y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x}$

6) $y'' + 9y = 0$
$m^2 + 9 = 0$
$m = \pm 3i$ → $y = c_1\cos 3x + c_2\sin 3x$

Undetermined Coefficients – Superposition Approach (4.4)

For non-homogeneous equations, the general solution is the sum of the "complementary" solution to the homogeneous equation (y_c) and a "particular" solution y_p. One way to find a particular solution is to guess, using the right side of the equation as a format guide.

If the right side is a:
polynomial: use a generic polynomial of the same degree
any constant → A
$3x$ → $Ax + B$
$5x^3$ → $Ax^3 + Bx^2 + Cx + D$

sine or cosine: use a generic sine <u>plus</u> cosine
$2\sin 3x$ → $A\sin 3x + B\cos 3x$

e^{mx}: use a generic e^{mx}
$3e^{3x}$ → Ae^{3x}

These forms may be added and multiplied:

$5x\sin 4x$ → $(Ax + B)\sin 4x + (Cx + D)\cos 4x$

$x^2e^x\cos 2x$ → $(Ax^2 + Bx + C)e^x\cos 2x + (Dx^2 + Ex + F)e^x\sin 2x$

If your guess for the particular solution duplicates something in the homogeneous solution, <u>multiply it by whatever power of x you need to create a non-duplicate</u>.
For example, if $y_c = c_1e^x + c_2xe^x$ and the function on the right side of your equation is $5e^x$, then the guess for y_p should be Ax^2e^x, not Ae^x.

<u>Examples</u>

1) $y'' + 4y = \mathbf{3\sin 2x}$ for $m^2 + 4 = 0$, $y_c = c_1\cos 2x + c_2\sin 2x$

$3\sin 2x$ suggests the form $A\cos 2x + B\sin 2x$, but that duplicates y_c, so multiply by x:

$$y_p = Ax\cos 2x + Bx\sin 2x$$
$$y_p' = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x$$

$$\boxed{\begin{array}{ll} y_p'' = & -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x \\ +\,4y_p: & \underline{\qquad\qquad + 4Ax\cos 2x \qquad\qquad + 4Bx\sin 2x\quad} \end{array}}$$

$$\mathbf{3\sin 2x} = -4A\sin 2x + 4B\cos 2x$$
$$A = -3/4 \qquad B = 0$$

$$y_p = -\frac{3}{4}x\cos 2x$$

general solution: $y = c_1\cos 2x + c_2\sin 2x - \dfrac{3}{4}x\cos 2x$

2) $y'' + 2y' - 24y = \mathbf{16 - (x + 2)e^{4x}}$ for $m^2 + 2m - 24 = 0$, $y_c = c_1e^{-6x} + c_2e^{4x}$

$\mathbf{16 - (x + 2)e^{4x}}$ suggests the form $A - (Bx + C)e^{4x}$, but that would duplicate the c_2e^{4x} term, so multiply the e^{4x} term by x:

$$y_p = A - (Bx + C)xe^{4x} = A - Bx^2e^{4x} - Cxe^{4x}$$
$$y_p' = -2Bxe^{4x} - 4Bx^2e^{4x} - Ce^{4x} - 4Cxe^{4x}$$

$$\begin{array}{rllllll} y_p'' = & -2Be^{4x} & - 16Bxe^{4x} & - 16Bx^2e^{4x} & - 8Ce^{4x} & - 16Cxe^{4x} & \\ +2y_p': & & -4Bxe^{4x} & - 8Bx^2e^{4x} & - 2Ce^{4x} & - 8Cxe^{4x} & \\ -24y_p: & & & + 24Bx^2e^{4x} & & + 24Cxe^{4x} & - 24A \\ \hline \end{array}$$

$\mathbf{16 - xe^{4x} - 2e^{4x}} = -2Be^{4x} - 10Ce^{4x} - 20Bxe^{4x} - 24A \quad \rightarrow \quad A = -\frac{2}{3} \quad B = \frac{1}{20} \quad C = \frac{19}{100}$

$$y_p = -\frac{2}{3} - \frac{1}{20}x^2e^{4x} - \frac{19}{100}xe^{4x}$$

general solution: $y = c_1e^{-6x} + c_2e^{4x} - \frac{2}{3} - \frac{1}{20}x^2e^{4x} - \frac{19}{100}xe^{4x}$

3) $y'' + y = \mathbf{2x\sin x}$ for $m^2 + 1 = 0$, $y_c = c_1\cos x + c_2\sin x$

$\mathbf{2x\sin x}$ suggests the form $(Ax + B)\sin x + (Cx + D)\cos x$, but that would duplicate the y_c terms, so multiply cosx and sinx by x:

$$y_p = (Ax + B)x\sin x + (Cx + D)x\cos x = Ax^2\sin x + Bx\sin x + Cx^2\cos x + Dx\cos x$$
$$y_p' = 2Ax\sin x + Ax^2\cos x + B\sin x + Bx\cos x + 2Cx\cos x - Cx^2\sin x + D\cos x - Dx\sin x$$

$$\begin{array}{rl} y_p'' = & 2A\sin x + 4Ax\cos x - Ax^2\sin x + 2B\cos x - Bx\sin x + 2C\cos x - 4Cx\sin x - Cx^2\cos x - 2D\sin x - Dx\cos x \\ + y_p: & \qquad + Ax^2\sin x \qquad + Bx\sin x \qquad + Cx^2\cos x \qquad + Dx\cos x \\ \hline \end{array}$$

$\mathbf{2x\sin x} = 2A\sin x + 4Ax\cos x + 2B\cos x + 2C\cos x - 4Cx\sin x - 2D\sin x$

$\rightarrow \quad A = 0 \qquad B = \frac{1}{2} \qquad C = -\frac{1}{2} \qquad D = 0$

$$y_p = -\frac{1}{2}x^2\cos x + \frac{1}{2}x\sin x$$

general solution: $y = c_1\cos x + c_2\sin x - \frac{1}{2}x^2\cos x + \frac{1}{2}x\sin x$

Undetermined Coefficients – Annihilator Approach (4.5)

Instead of using the "m" notation in Section 4.3, we can use a differential operator "D" notation to indicate an operator that annihilates a function, or makes it zero. D^2 annihilates x because the second derivative of x is zero.

For $y'' + 3y' + 2y = 0$, $m^2 + 3m + 2 = 0$ is the auxiliary equation. In annihilator notation, $(D^2 + 3D + 2)y = 0$. <u>The change in notation doesn't affect the format of the solutions (see 4.3)</u>. For this example, $y_c = c_1e^{-x} + c_2e^{-2x}$.

To find particular solutions, the following annihilators are used:

annihilator	annihilates *any linear combination* of
D^n	$1, x, x^2, x^3, \ldots, x^{n-1}$
$(D - \alpha)^n$	$e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}, \ldots, x^{n-1}e^{\alpha x}$
$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$	$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, x^2e^{\alpha x}\cos\beta x, \ldots, x^{n-1}e^{\alpha x}\cos x$ *and* $e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, x^2e^{\alpha x}\sin x, \ldots, x^{n-1}e^{\alpha x}\sin x$

Combining the annihilator of the "right-side function" of a non-homogeneous equation with the annihilator of the homogeneous equation produces the general solution.

Finding general solutions

1. Find the annihilator for the homogenous equation and write in D notation.
2. Multiply both sides of the equation by the annihilator for the right-side function.
3. A product of annihilators annihilates a sum of solution components.
4. Ignore the y_c components and use the rest to solve the equation for the particular solution.

<u>Examples</u>

1) $y'' + 3y' + 2y = 2x^2$ $\qquad (m + 2)(m + 1) = 0 \quad \rightarrow \quad y_c = c_1e^{-2x} + c_2e^{-x}$

$(D + 2)(D + 1)$ annihilates the left side and D^3 annihilates the right side. So:

$$D^3(D + 2)(D + 1)y = D^3(2x^2) = 0$$

The auxiliary equation is $m^3(m + 2)(m + 1) = 0$ and its roots are $m = -2, -1, 0, 0, 0$

general solution: $y = \underbrace{c_1e^{-2x} + c_2e^{-x}}_{y_c} + \underbrace{c_3 + c_4x + c_5x^2}_{y_p}$ (use A, B, C for convenience for y_p)

$y_p = A + Bx + Cx^2$

$y_p' = B + 2Cx$

$$\begin{array}{rl} y_p'' = & 2C \\ + 3y_p': & +3B + 6Cx \\ + 2y_p: & +2A + 2Bx + 2Cx^2 \\ \hline \end{array}$$

$2x^2 = 2Cx^2 + 6Cx + 2Bx + 2C + 3B + 2A$

$C = 1 \qquad B = -3 \qquad A = 7/2$

general solution: $y = c_1e^{-2x} + c_2e^{-x} + 7/2 - 3x + x^2$

2) $y'' - 2y' + 5y = e^{2x}\sin x$ $(m^2 - 2m + 5) = 0 \rightarrow y_c = c_1e^x\cos 2x + c_2\, e^x\sin 2x$

$(D^2 - 2D + 5)$ annihilates the left side. On the right side, $\alpha = 2$, $\beta = 1$, and the annihilator is $[D^2 - 2(2)D + (2^2 + 1^2)] = (D^2 - 4D + 5)$. So:

$$(D^2 - 4D + 5)\,(D^2 - 2D + 5)y = (D^2 - 4D + 5)(\,e^{2x}\sin x) = 0$$

The auxiliary equation is $(m^2 - 4m + 5)\,(m^2 - 2m + 5) = 0$ and the roots are $1 \pm 2i$ and $2 \pm i$.

general solution: $$y = \underbrace{c_1e^x\cos 2x + c_2\, e^x\sin 2x}_{y_c} + \underbrace{c_3e^{2x}\cos x + c_4e^{2x}\sin x}_{y_p}$$

$$\begin{aligned} y_p &= Ae^{2x}\cos x + Be^{2x}\sin x \\ y_p' &= 2Ae^{2x}\cos x - Ae^{2x}\sin x + 2Be^{2x}\sin x + Be^{2x}\cos x \end{aligned}$$

$$\begin{array}{ll} y_p'' = & 3Ae^{2x}\cos x - 4Ae^{2x}\sin x + 3Be^{2x}\sin x + 4Be^{2x}\cos x \\ -2y_p': & -4Ae^{2x}\cos x + 2Ae^{2x}\sin x - 4Be^{2x}\sin x - 2Be^{2x}\cos x \\ +5y_p: & +5Ae^{2x}\cos x \qquad\qquad\qquad + 5Be^{2x}\sin x \\ \hline \end{array}$$

$$e^{2x}\sin x = 4Ae^{2x}\cos x - 2Ae^{2x}\sin x + 4Be^{2x}\sin x + 2Be^{2x}\cos x$$

$$\left.\begin{aligned} -2A + 4B &= 1 \\ 4A + 2B &= 0 \end{aligned}\right\} \rightarrow \quad A = -\frac{1}{10} \qquad B = \frac{1}{5}$$

general solution: $$y = c_1e^x\cos 2x + c_2\, e^x\sin 2x - \frac{1}{10}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x$$

3) $y'' + y = \cos x - 4\sin x$ $(m^2 + 1) = 0 \rightarrow y_c = c_1\cos x + c_2\sin x$

$(D^2 + 1)$ annihilates the left side and $(D^2 + 1)$ also annihilates the right side ($\alpha = 0$, $\beta = 1$). So:

$$(D^2 + 1)(D^2 + 1)y = (D^2 + 1)(\cos x - 4\sin x) = 0$$

The auxiliary equation is $(m^2 + 1)^2 = 0$ and the roots are $\pm i$, $\pm i$.

general solution: $$y = \underbrace{c_1\cos x + c_2\sin x}_{y_c} + \underbrace{c_3x\cos x + c_4x\sin x}_{y_p}$$

$$\begin{aligned} y_p &= Ax\cos x + Bx\sin x \\ y_p' &= A\cos x - Ax\sin x + B\sin x + Bx\cos x \end{aligned}$$

$$\begin{array}{ll} y_p'' = & -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x \\ +y_p: & \qquad\qquad + Ax\cos x \qquad\qquad + Bx\sin x \\ \hline \end{array}$$

$$\cos x - 4\sin x = 2B\cos x - 2A\sin x \qquad \rightarrow \qquad A = 2 \qquad B = \frac{1}{2}$$

general solution: $$y = c_1\cos x + c_2\sin x + 2x\cos x + \frac{1}{2}x\sin x$$

4) $y''' - 3y'' + 3y' - y = e^x - 2x + 4$ $(m-1)^3 = 0 \rightarrow y_c = c_1e^x + c_2xe^x + c_3x^2e^x$

$(D-1)^3$ annihilates the left side and $(D-1)D^2$ annihilates the right side (remember, a product of annihilators annihilates a sum of solution components). So:

$$D^2(D-1)(D-1)^3y = D^2(D-1)(e^x - 2x + 4) = 0$$

The auxiliary equation is $m^2(m-1)^4 = 0$ and the roots are 1, 1, 1, 1, 0, 0

general solution: $$y = \underbrace{c_1e^x + c_2xe^x + c_3x^2e^x}_{y_c} + \underbrace{c_4x^3e^x + c_5 + c_6x}_{y_p}$$

$$\begin{aligned} y_p &= Ax^3e^x + Cx + B \\ y_p' &= 3Ax^2e^x + Ax^3e^x + C \\ y_p'' &= 6Axe^x + 6Ax^2e^x + Ax^3e^x \end{aligned}$$

$$\boxed{\begin{array}{ll} y_p''' = & 6Ae^x + 18Axe^x + 9Ax^2e^x + Ax^3e^x \\ -3y_p'': & \quad -18Axe^x - 18Ax^2e^x - 3Ax^3e^x \\ +3y_p': & \qquad\qquad +9Ax^2e^x + 3Ax^3e^x + 3C \\ -y_p: & \qquad\qquad\qquad -Ax^3e^x - Cx - B \\ \hline \end{array}}$$

$e^x - 2x + 4 = 6Ae^x - Cx + (3C - B)$ $\rightarrow$ $A = \dfrac{1}{6}$ $C = 2$ $B = 2$

general solution: $$y = c_1e^x + c_2xe^x + c_3x^2e^x + \frac{1}{6}x^3e^x + 2x + 2$$

Variation of Parameters (4.6)

The variation of parameters method of solving a second-order equation uses the standard form:

$$y'' + P(x)y' + Q(x)y = f(x).$$

Using y_1 and y_2 from the homogeneous equation solution set, finding a particular solution assumes:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Method

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2f(x) \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1f(x)$$

$$u_1' = \frac{W_1}{W} = \frac{-y_2f(x)}{W} \qquad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W} \qquad \to \text{ Integrate } u_1' \text{ and } u_2'.$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

For third-degree $y''' + P_2(x)y'' + P_1(x)y' + P_0(x)y = f(x)$:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad W_1 = f(x)\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} \qquad W_2 = -f(x)\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} \qquad W_3 = f(x)\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W} \qquad u_3' = \frac{W_3}{W} \qquad \to \text{ Integrate } u_1',\ u_2', \text{ and } u_3'$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x)$$

Examples

1) $3y'' - 6y' + 6y = 3e^x\csc x$

$y'' - 2y' + 2y = e^x\csc x$ $\qquad m^2 - 2m + 2 = 0 \qquad m = 1 \pm i \;\to\; y_c = c_1e^x\cos x + c_2e^x\sin x$

$y_1 = e^x\cos x \qquad y_2 = e^x\sin x$

$$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\sin x + e^x\cos x \end{vmatrix} = e^{2x}$$

$$W_1 = -y_2f(x) = -e^x\sin x(e^x\csc x) = -e^{2x} \qquad W_2 = y_1f(x) = e^x\cos x(e^x\csc x) = e^{2x}\cot x$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{2x}}{e^{2x}} = -1 \qquad u_2' = \frac{W_2}{W} = \frac{e^{2x}\cot x}{e^{2x}} = \cot x$$

$u_1 = -x \qquad u_2 = \ln|\sin x| \qquad y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = -xe^x\cos x + e^x\sin x\ln|\sin x|$

general solution: $y = c_1e^x\cos x + c_2e^x\sin x - xe^x\cos x + e^x\sin x\ln|\sin x|$

2) $y'' + y = \cot x$ $\qquad m^2 + 1 = 0 \qquad m = \pm i \;\rightarrow\; y_c = c_1\cos x + c_2\sin x$

$y_1 = \cos x \qquad y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = -y_2 f(x) = -\sin x \cot x = -\cos x \qquad W_2 = y_1 f(x) = \cos x \cot x = \frac{\cos^2 x}{\sin x} = \csc x - \sin x$$

$$u_1' = W_1 = -\cos x \qquad u_2' = W_2 = \csc x - \sin x$$

$$u_1 = \int -\cos x\,dx = -\sin x \qquad u_2 = \int (\csc x - \sin x)dx = \ln|\csc x - \cot x| + \cos x$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = -\sin x\cos x + \sin x[\ln|\csc x - \cot x| + \cos x] = \sin x[\ln|\csc x - \cot x|]$$

general solution: $y = c_1\cos x + c_2\sin x + \sin x[\ln|\csc x - \cot x|]$

3) $y''' + y' = \cot x$ $\qquad m^3 + m = m(m^2 + 1) \quad m = 0, \pm i \quad \rightarrow \quad y_c = c_1 + c_2\cos x + c_3\sin x$

$y_1 = 1 \qquad y_2 = \cos x \qquad y_3 = \sin x$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = 1$$

$$u_1' = W_1 = \cot x\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cot x$$

$$u_2' = W_2 = -\cot x\begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} = -\cos x\cot x = \sin x - \csc x$$

$$u_3' = W_3 = \cot x\begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = -\sin x\cot x = -\cos x$$

$$u_1 = \int \cot x\,dx = \ln|\sin x| \qquad u_2 = \int (\sin x - \csc x)dx = -\cos x - \ln|\csc x - \cot x|$$

$$u_3 = \int -\cos x\,dx = -\sin x$$

$$y_p = \ln|\sin x| - \cos x[\cos x + \ln|\csc x - \cot x|] - \sin x(\sin x)$$

general solution: $y = c_1 + c_2\cos x + c_3\sin x + \ln|\sin x| - \cos x[\cos x + \ln|\csc x - \cot x|] - \sin^2 x$