Matrix and Linear Transformations

Functions and transformations are nearly identical in concept. While functions in Algebra manipulate the real numbers, transformations in Linear Algebra manipulate the vectors in vector spaces. One key difference, though, is that while functions are restricted to $\mathbf{R}$, transformations are not. (Remember, $\mathbf{R}^n$ is the set of sequences of n real numbers.)

Recall from Algebra that a function inputs x values and outputs y values. For example, let's say we have a function where $f(2) = 5$. The input is $x = 2$ and the output is $y = 5$, or we can say that the *image* of 2 is 5. Likewise, a transformation <u>assigns a vector from one set into a unique vector in another set</u>. For example, consider the transformation T of $\mathbf{R}^3$ into $\mathbf{R}^2$ defined by $T(x, y, z) = (2x, y - z)$. Let $\mathbf{u} = (-1, 5, 3)$ be some vector in $\mathbf{R}^3$. We can find the image of $\mathbf{u}$ by setting $x = -1$, $y = 5$, and $z = 3$ in T. We get $T(-1, 5, 3) = (-2, 2)$. Therefore, the *image* of $(-1, 5, 3)$ is $(-2, 2)$.

Note that $T(-1, 5, 3) = (-2, 2)$ can also be written as $T\left(\begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. It is often convenient to write vectors in column form while using transformations.

In Algebra, we discuss the domain and range of functions. For transformations, we use the *codomain* instead of the range (or image) of the transformation. (In the example above, the domain of T is $\mathbf{R}^3$ and the codomain is $\mathbf{R}^2$). Why focus on the codomain and not the range? Observe that the range of T is the set of all ordered pairs of the form $(2x, y - z)$. We know the *pattern* of the range, but we don't know the *size* of it. Is it large or small? Does the range use all of the real numbers or only a subset? The only thing that we do know is that the images of T are elements of $\mathbf{R}^2$, the codomain in this case. Therefore, it is easier to focus on the codomain than to try to figure out the range of the transformation every time. In general, the range of T is always a subset of the codomain.

Definition: A *transformation* T of $\mathbf{R}^n$ into $\mathbf{R}^m$ assigns each vector $\mathbf{u}$ in $\mathbf{R}^n$ a unique vector $\mathbf{v}$ in $\mathbf{R}^m$, with $\mathbf{R}^n$ the *domain* and $\mathbf{R}^m$ the *codomain*. We write $T(\mathbf{u}) = \mathbf{v}$. The term *mapping* is also used for transformations. So, we can say either $\mathbf{v}$ is the *image* of $\mathbf{u}$, or T *maps* $\mathbf{u}$ *into* $\mathbf{v}$.

We will now present a few transformations that are very similar to function translations in Algebra.

Dilation and Contraction

Consider the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix}$, where r is a positive scalar. T maps every ordered pair in $\mathbf{R}^2$ r times closer to, or farther away from, the origin. If $r > 1$, then the ordered pair moves farther away from the origin and is called a *dilation of factor r.* If $0 < r < 1$, then the ordered pair moves closer to the origin and is called a *contraction of factor r.* This transformation can also be written in matrix form:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = r\begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Example 1: When $r = 3$, T maps an ordered pair three times as far away from the origin. When $r = ½$, T maps an ordered pair one-half its distance from the origin. If we have an ordered pair (-2, 4) for example, then the images from these two transformations will be (-6, 12) and (-1, 2), respectively.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}, \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ½ & 0 \\ 0 & ½ \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}$$

Example 2: Find the equation of the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 4.

In this example, $r = 4$. Since $4 > 1$, the ordered pairs of the unit circle move four times away from the origin. This scales the radius of the unit circle from 1 to 4. So, the equation of the image is $x^2 + y^2 = 16$.

Reflection

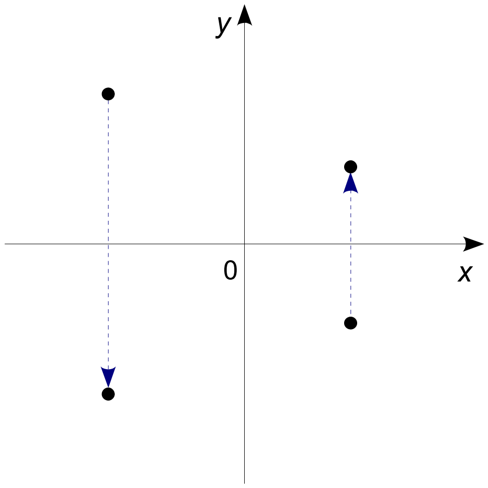
Transformations can also reflect ordered pairs across the x- and y-axes. Consider the transformation

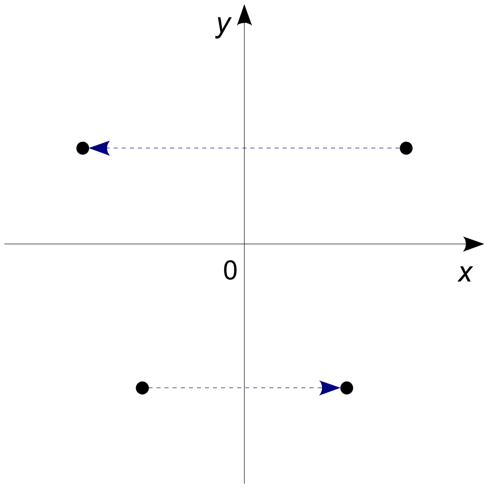
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$. T reflects every ordered pair in $\mathbf{R}^2$ across the x-axis, whereas the transformation

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}$ reflects across the y-axis. Why do these work? Recall from Algebra that when an ordered pair is reflected across the x-axis, the sign of the y component changes. Likewise, when reflecting across the y-axis, the sign of the x component changes. (With this in mind, what would be the transformation that would reflect ordered pairs across the origin?)

As we have seen with dilation and contraction, reflection transformations can also be written in matrix form.

$$x\text{-axis} \rightarrow T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad y\text{-axis} \rightarrow T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$





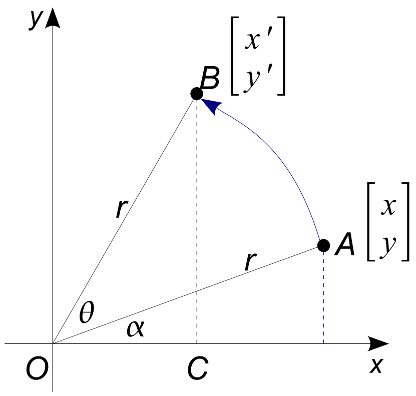
Example 3: Find the image of $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ under the reflection transformation across the y-axis.

$$T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

Rotation about the Origin

This one may be a little tricky to see at first. The transformation for rotation about the origin is

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, where θ is the angle of rotation (positive for counterclockwise and negative for clockwise). Why does this work? Let's derive it and find out (next page).



Since the two vectors **OA** and **OB** are the same length, let r = **OA** = **OB**. From trigonometry, we know that $x = r\cos(\alpha)$ and $y = r\sin(\alpha)$. So,

$$x' = r\cos(\alpha+\theta) = r\cos(\alpha)\cos(\theta) - r\sin(\alpha)\sin(\theta) = x\cos(\theta) - y\sin(\theta) \text{ and}$$

$$y' = r\sin(\alpha+\theta) = r\sin(\alpha)\cos(\theta) + r\sin(\theta)\cos(\alpha) = y\cos(\theta) + x\sin(\theta).$$

Written in matrix form, we get:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Example 4: Determine the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ with a rotation of $\frac{\pi}{2}$ about the origin.

Plugging $\frac{\pi}{2}$ in for θ, we obtain the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. So, the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Example 5: Find the equation of the image of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ under a rotation through an angle of $\frac{\pi}{2}$.

We will need to approach this problem a little differently than in Example 4. From the proof of the rotation transformation, we know that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Since $\theta = \frac{\pi}{2}$, this becomes $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. From this we obtain $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y' \\ -x' \end{bmatrix}$, which gives us x and y to plug into the ellipse. So, the equation of the image is

$$\frac{(y')^2}{4} + \frac{(-x')^2}{9} = 1 \rightarrow \frac{(x')^2}{9} + \frac{(y')^2}{4} = 1 .$$

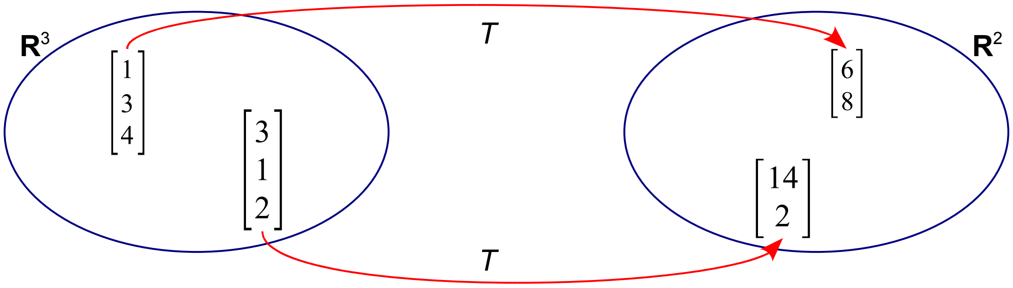
Matrix Transformations

Observe that we were able to define dilation, contraction, reflection, and rotation using matrices. Every matrix can in fact define a transformation. Let A be a matrix and $\mathbf{x}$ a column vector such that $A\mathbf{x}$ exists. Then A defines the *matrix transformation* $T(\mathbf{x}) = A\mathbf{x}$.

Definition: Let A be a $m \times n$ matrix. Let $\mathbf{x}$ be an element of $\mathbf{R}^n$ written in column matrix form. A defines a *matrix transformation* $T(\mathbf{x}) = A\mathbf{x}$ of $\mathbf{R}^n$ into $\mathbf{R}^m$, with *domain* $\mathbf{R}^n$ and *codomain* $\mathbf{R}^m$. T *maps* $\mathbf{R}^n$ into $\mathbf{R}^m$, denoted by $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$, and the *image* of $\mathbf{x}$ is the vector $A\mathbf{x}$.

Example 6: Let $A = \begin{bmatrix} 5 & 3 & -2 \\ 0 & 4 & -1 \end{bmatrix}$. Then, A defines the transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5 & 3 & -2 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

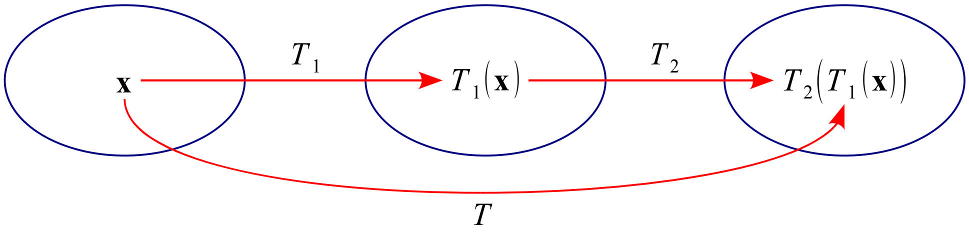
We find the image of a vector such as $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ to be $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$, denoted as $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \longmapsto \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Likewise, we can choose another vector $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and find its image to be $T\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 & 3 & -2 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$.



Composition of Transformations

Composition of transformations can be thought of exactly like composition of functions. Consider the matrix transformations $T_1(\mathbf{x}) = A_1\mathbf{x}$ and $T_2(\mathbf{x}) = A_2\mathbf{x}$, where A_1 and A_2 are $m \times n$ matrices. The composite transformation T is given by:

$$T = T_2 \circ T_1 \;\rightarrow\; T(\mathbf{x}) = T_2\left(T_1(\mathbf{x})\right) = T_2\left(A_1\mathbf{x}\right) = A_2 A_1 \mathbf{x}$$



Naturally, we can extend this concept even further. Let $T_1, \ldots, T_n$ be a sequence of n transformations defined by the matrices $A_1, \ldots, A_n$. Then, the composite transformation $T = T_n \circ T_{n-1} \circ \cdots \circ T_1$ is defined by the matrix product $A_n \cdot A_{n-1} \cdot \ldots \cdot A_1$ (assuming it exists).

Example 7: Determine the transformation that describes a reflection in the *x*-axis, followed by a rotation through $\frac{\pi}{2}$, followed by a dilation of factor 3. Find the image of the ordered pair $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let F be the reflection matrix, R be the rotation matrix, and D be the dilation matrix. From our discussion earlier, we know that $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Since composition follows a right-to-left order, our transformation is

$T(\mathbf{x}) = D \cdot R \cdot F(\mathbf{x}) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}(\mathbf{x})$. The image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

Linear Transformations

We have discussed the uses of a matrix transformation; now let's discuss the properties. Recall that a vector space has the two operations of addition and scalar multiplication. How would the matrix transformation $T(\mathbf{u}) = A(\mathbf{u})$ interact with these operations? Let *c* be a scalar and let **u** and **v** be two vectors in the domain of *T*. By the matrix properties of *A*,

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = T(\mathbf{u})+T(\mathbf{v}) \text{ and } T(c\,\mathbf{u}) = A(c\,\mathbf{u}) = c\,A\,\mathbf{u} = c\,T(\mathbf{u}) .$$

This implies that *T* preserves vector addition and scalar multiplication. That is, *T preserves vector space structure*. We call such transformations with these properties *linear transformations*.

Definition: A transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is said to be *linear* if *T* preserves vector addition and scalar multiplication. <u>Every matrix transformation is linear</u>.

Example 8: Consider the transformation $T(x, y) = (x+y, 4\text{x})$. Show that *T* is a linear transformation.

Let (x_1, y_1) and (x_2, y_2) be two elements in $\mathbf{R}^2$. Let *c* be a scalar. Then,

$$\begin{aligned} T\big((x_1, y_1) + (x_2, y_2)\big) &= T(x_1+x_2,\ y_1+y_2) = \big(x_1+x_2+y_1+y_2,\ 4(x_1+x_2)\big) \\ &= (x_1+y_1, 4x_1) + (x_2+y_2, 4x_2) = T(x_1, y_1) + T(x_2, y_2) \end{aligned}$$

and

$$T\big(c(x_1, y_1)\big) = T(c\,x_1, c\,y_1) = \big(c\,x_1+c\,y_1, 4(c\,x_1)\big) = c(x_1+y_1, 4x_1) = c\,T(x_1, y_1) .$$

Since *T* preserves vector addition and scalar multiplication, *T* is linear.

Example 9: Show that $T(x, y, z) = (x^2, y\,z)$ is not a linear transformation.

Let *c* be a scalar. Then,

$$T\big(c(x, y, z)\big) = T(c\,x, c\,y, c\,z) = \big((c\,x)^2, (c\,y)(c\,z)\big) = c^2(x^2, yz) = c^2\,T(x, y, z) \neq c\,T(x, y, z) .$$

Since *T* does not preserve scalar multiplication, *T* is not a linear transformation.

Notice that in order to show a transformation is linear, we need to prove both properties are true (Example 8). However, to show a transformation is not linear, we only need to prove that one of the properties does not hold true (Example 9). <u>Both properties need to be true in order for the transformation to be linear</u>.

Matrix Representation

Previously, we have used static, problem-by-problem methods to derive matrices to represent rotations, dilations, and reflections. Now we will use a method to construct a matrix representation for any linear transformation on $\mathbf{R}^n$. Consider the following example:

Example 10: Determine a matrix A that describes the linear transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2\text{x}+y \\ 3\text{y} \end{bmatrix}$.

Notice that T is linear (prove it), with a domain of $\mathbf{R}^2$. Applying T on the standard basis of $\mathbf{R}^2$, we get $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The images of the standard basis vectors become the column vectors of our matrix A. So, $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Now to see why this method works.

General Result:

Let T be a linear transformation on $\mathbf{R}^n$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbf{R}^n$ and $\mathbf{u}$ be an arbitrary vector in $\mathbf{R}^n$. In row form,

$$\mathbf{e}_1 = (1,0, \dots ,0),\ \mathbf{e}_2 = (0,1,0, \dots ,0),\ \cdots ,\ \mathbf{e}_n = (0,0, \dots ,1),\ \text{and}\ \ \mathbf{u} = (c_1, c_2, \dots , c_n).$$

Notice that we can express $\mathbf{u}$ in terms of the standard basis: $\mathbf{u} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n$.

Since T is a linear transformation,

$$T(\mathbf{u}) = T(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n) = c_1 T(\mathbf{e}_1) + \cdots + c_n T(\mathbf{e}_n) = \left[T(\mathbf{e}_1) \dots T(\mathbf{e}_n)\right]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

where $\left[T(\mathbf{e}_1) \dots T(\mathbf{e}_n)\right]$ is a matrix with column vectors $T(\mathbf{e}_1), \dots , T(\mathbf{e}_n)$. Therefore, the linear transformation T is defined by the matrix $A = \left[T(\mathbf{e}_1) \dots T(\mathbf{e}_n)\right]$, called the *standard matrix* of T.

Example 11: Determine the standard matrix of the linear transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4x+3y \\ x-2y \end{bmatrix}$. Find the image of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Applying the transformation on the standard basis of $\mathbf{R}^2$, we get

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. This gives us the standard matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$. So, the transformation can be written as $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Applying the transformation to $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, we find the image to be $\begin{bmatrix} 16 \\ -7 \end{bmatrix}$.

Exercises:

1) Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ under a dilation of factor 5.

2) Find the equation of the image of the circle $x^2+y^2=4$ under a contraction of factor ½.

3) Determine the matrix that defines a reflection in $y=x$. Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ under this transformation.

4) Determine the image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under a rotation of $\frac{\pi}{4}$ about the origin

5) Find the equation of the image of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ under a rotation through $\theta = \frac{3\pi}{2}$.

6) Consider the transformation T defined by $A = \begin{bmatrix} 2 & 6 & 4 \\ 0 & -2 & 3 \end{bmatrix}$. Find the image of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

7) Let $T = T_2 \circ T_1$ and let $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$. Find the matrix that defines T and determine the image of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

8) Consider the transformation $T(x, y) = (3x-2y, 5y)$. Is T linear?

9) Consider the transformation $T(x, y, z) = (x+3z, 6y^2)$. Is T linear?

10) Find the standard matrix of the linear transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x-y \end{bmatrix}$.

11) Find the standard matrix of the linear transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z+2x \\ z+3y \end{bmatrix}$.

Answers:

1) $\begin{bmatrix} 15 \\ 10 \end{bmatrix}$	2) $x^2 + y^2 = 1$	3) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
4) $\begin{bmatrix} -1/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}$	5) $\frac{(x')^2}{9} + \frac{(y')^2}{4} = 1$	6) $\begin{bmatrix} 14 \\ 9 \end{bmatrix}$
7) $\begin{bmatrix} -1 & -2 \\ 16 & 2 \end{bmatrix}$, $\begin{bmatrix} -12 \\ 42 \end{bmatrix}$	8) Yes	9) No
10) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$	11) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$	