LAPLACE TRANSFORMS - TRANSLATION THEOREMS

By definition the Laplace transform of f(t) is

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s).$$

Some of the most common transforms are:

$$L\{1\} = \frac{1}{s} \qquad L\{t^n\} = \frac{n!}{s^{n+1}} \qquad L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{Sinkt\} = \frac{k}{s^2+k^2} \qquad L\{Sinhkt\} = \frac{k}{s^2-k^2}$$

$$L\{Coskt\} = \frac{s}{s^2+k^2} \qquad L\{Coshkt\} = \frac{s}{s^2-k^2}$$

These are obtained from the definition by integrating by parts. In each of these transforms the expression in the bracket is f(t), and the result on the right side is F(s).

If we seek the Laplace transform of $e^{at} f(t)$ we find

$$L\{e^{at} f(t)\} = \int e^{-(s-a)t} f(t)dt = F(s-a).$$

This result is known as the First Translation Theorem, so called because F(s-a) translates the graph of F(s) "a" units to the right. The notation for this is

$$L\{e^{at} f(t)\} = L\{f(t)\}\big|_{s \to s-a} = F(s)\big|_{s \to s-a} = F(s-a)$$, where $s \to s-a$

means s is replaced by s-a.

<u>Example</u>: Consider $L\{e^{5t}t^6\}$. Here, a = 5 and $f(t) = t^6$. Then,

$$L\{e^{5t}t^6\} = L\{t^6\}|_{s\to s-5} = \frac{6!}{s^7}\Big|_{s\to s-5} = \frac{6!}{(s-5)^7}.$$

Example: Consider $L\{e^{-t}\cos 3t\}$. Here, a = -1 and f(t) = cos3t. Then,

$$L\{e^{-t}\cos 3t\} = L\{\cos 3t\}|_{s\to s+1} = \frac{s}{s^2+9}\Big|_{s\to s+1} = \frac{s+1}{(s+1)^2+9}.$$

By taking the inverse of a Laplace Transform, that is, $L^{-1}\{F(s)\}$, we would get back the original function, f(t). For example:

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \qquad L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$
$$L^{-1}\left\{\frac{k}{s^2+k^2}\right\} = Sinkt \qquad L^{-1}\left\{\frac{k}{s^2-k^2}\right\} = Sinhkt$$
$$L^{-1}\left\{\frac{s}{s^2+k^2}\right\} = Coskt \qquad L^{-1}\left\{\frac{s}{s^2-k^2}\right\} = Coshkt$$

Example: Find $L^{-1}\left\{\frac{1}{s^6}\right\}$.

$$L^{-1}\left\{\frac{1}{s^6}\right\} = \frac{1}{5!}L^{-1}\left\{\frac{5!}{s^6}\right\} = \frac{1}{5!}t^5 = \frac{t^5}{5!}$$

Example: Find $L^{-1}\left\{\frac{2s-6}{s^2-5}\right\}$

$$L^{-1}\left\{\frac{2s-6}{s^2-5}\right\} = L^{-1}\left\{\frac{2s}{s^2-5}\right\} - L^{-1}\left\{\frac{6}{s^2-5}\right\}$$
$$= 2L^{-1}\left\{\frac{s}{s^2-5}\right\} - \frac{6}{\sqrt{5}}L^{-1}\left\{\frac{\sqrt{5}}{s^2-5}\right\}$$
$$= 2Cosh\sqrt{5}t - \frac{6}{\sqrt{5}}Sinh\sqrt{5}t$$

Since $L\{e^{at}f(t)\} = L\{f(t)\}|_{s\to s-a} = F(s)|_{s\to s-a} = F(s-a)$, taking inverses we get

$L^{-1}\{F(s-a)\} = L^{-1}\{F(s)\}\big|_{s-a\to s} = e^{at} f(t).$ Here $s-a \to s$ means that

$s-a$ is replaced by s.

<u>Example</u>: Find $L^{-1}\left\{\dfrac{k}{(s+2)^2+k^2}\right\}$.

$$L^{-1}\left\{\frac{k}{(s+2)^2+k^2}\right\} = L^{-1}\left\{\frac{k}{s^2+k^2}\right\}_{s+2\to s} = e^{-2t} Sinkt$$

<u>Example</u>: Find $L^{-1}\left\{\dfrac{1}{(s-3)^4}\right\}$.

$$L^{-1}\left\{\frac{1}{(s-3)^4}\right\} = \frac{1}{3!} L^{-1}\left\{\frac{3!}{(s-3)^4}\right\} = \frac{1}{3!} L^{-1}\left\{\frac{3!}{s^4}\right\}_{s-3\to s} = \frac{e^{3t} t^3}{3!}$$

If f(t-a) is multiplied by the unit step function u(t-a), the transform is changed to $e^{-as}F(s)$: if g(t) is multiplied by the unit step function u(t-a), the transform is changed to $e^{-as}L\{g(t+a)\}$. That is,

$$L\ \{f(t-a)u(t-a)\} = e^{-as}F(s)$$
$$L\ \{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

This is known as the Second Translation Theorem.

<u>Example</u>: Evaluate $L\{(t-3)^2 u(t-3)\}$

Here $\quad a = 3,\ f(t) = t^3,\ F(s) = L\{t^2\} = \dfrac{2!}{s^3}$

$\text{Then, } L\{(t-3)^2\, u(t-3)\} = \dfrac{2!e^{-3s}}{s^3}.$

<u>Example</u>: Evaluate $L\{(t-3)^2 u(t-3)\}$

Here $\quad a = 3, g(t) = (t-3)^2, \text{so } g(t+3) = t^2$

$\text{Then, } L\{(t-3)^2\, u(t-3)\} = e^{-3s}L\{t^2\} = \dfrac{2e^{-3s}}{s^3}.$

The inverse form of the Second Translation Theorem is

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

<u>Example</u>: Find $L^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

$\frac{1}{2}L^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2{}_{t\to t-2} = \frac{1}{2}(t-2)^2$ so the answer is $\frac{1}{2}(t-2)^2 u(t-2)$.

Exercises:

1. $L\{e^{-3t}Sin2t\}$
2. $L\{e^t(t-3)^3\}$
3. $L\{(t-2)u(t-2)\}$
4. $L\{e^{3-t}u(t-3)\}$
5. $L\{(t-2)^3 e^{t-2}u(t-2)\}$
6. $L^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$
7. $L^{-1}\left\{\frac{e^{\frac{-\pi s}{2}}}{s^2+1}\right\}$

Answers:

1. $\frac{2}{(s+3)^2+4}$
2. $\frac{3!}{(s-1)^4} - \frac{12}{(s-1)^3} + \frac{12}{(s-1)^2} - \frac{8}{(s-1)}$
3. $\frac{e^{-2s}}{s^2}$
4. $\frac{e^{-3s}}{s+1}$
5. $\frac{6e^{-s}}{(s-1)^4}$
6. $u(t-1) - e^{-(t-1)}u(t-1)$
7. $(-Cost)u(t-\frac{\pi}{2})$