Factorial Analysis of Variance (ANOVA)

Previously, with single-factor ANOVA, we only had one variable (or factor) of interest. However, what if in our experiment we had more than one variable of interest? We would need a multiple-factor (or **factorial**) **Analysis of Variance**. For our purpose, we will focus on the two-way ANOVA. In a two-factor ANOVA test, it is common to organize the data in a table, and to label one of the factors as the row factor and the other as the column factor. Here are the hypotheses for this ANOVA:

1. H_0: There is no difference in population means among the levels of the row factor.
   H_1: At least two population means are different among the levels of the row factor.

2. H_0: There is no difference in population means among the levels of the column factor.
   H_1: At least two population means are different among the levels of the column factor.

3. H_0: There is no interaction between the factors.
   H_1: There is an interaction between the factors.

Notice that not only are we comparing the population means of the row and column factors, but we are also testing to see if there is *any interaction* at all between these two factors.

Below is the general form of the two-way ANOVA table.

Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	***F***	***p*-value**
Factor A	SSA	$a-1$	$\text{MSA}=\frac{\text{SSA}}{a-1}$	$\frac{\text{MSA}}{\text{MSE}}$	
Factor B	SSB	$b-1$	$\text{MSB}=\frac{\text{SSB}}{b-1}$	$\frac{\text{MSB}}{\text{MSE}}$	
Interaction	SSAB	$(a-1)(b-1)$	$\text{MSAB}=\frac{\text{SSAB}}{(a-1)(b-1)}$	$\frac{\text{MSAB}}{\text{MSE}}$	
Error	SSE	$ab(r-1)$	$\text{MSE}=\frac{\text{SSE}}{ab(r-1)}$		
Total	SST	n_T-1			

In this table, a is the number of levels of Factor A, b is the number of levels of Factor B, r is the number of replications (sample size per treatment), and n_T is the total number of observations from the experiment. (In Excel, if you have replications, select "ANOVA: Two-Factor with Replication"). To compute the values required for the two-way ANOVA table, we first need the following definitions:

1. x_{ijk} = observation corresponding to the k^{th} replicate taken from treatment i of factor A and treatment j of factor B
2. $\bar{x}_{i\cdot}$ = sample mean of the observations in treatment i in factor A
3. $\bar{x}_{\cdot j}$ = sample mean of the observations in treatment j in factor B

4. $\bar{x}_{ij}$ = sample mean of the observations corresponding to treatment *i* of factor A and treatment *j* of factor B
5. $\bar{\bar{x}}$ = overall sample mean of all n_T observations

To help keep everything in order, we use the following steps to compute the Sum of Squares column.

1. Compute the total sum of squares: $\text{SST} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\left(x_{ijk} - \bar{\bar{x}}\right)^2$
2. Compute the sum of squares for factor A: $\text{SSA} = br\sum_{i=1}^{a}\left(\bar{x}_{i\cdot} - \bar{\bar{x}}\right)^2$
3. Compute the sum of squares for factor B: $\text{SSB} = ar\sum_{j=1}^{b}\left(\bar{x}_{\cdot j} - \bar{\bar{x}}\right)^2$
4. Compute the sum of squares for interaction: $\text{SSAB} = r\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{x}_{ij} - \bar{x}_{i\cdot} - \bar{x}_{\cdot j} + \bar{\bar{x}}\right)^2$
5. Compute the sum of squares due to error: $\text{SSE} = \text{SST} - \text{SSA} - \text{SSB} - \text{SSAB}$

Example:
The Graduate Management Admissions Test (GMAT) is a standardized test used to evaluate an applicant's ability to perform in a field of business. Suppose we want to analyze the effects of three different test review strategies on the GMAT scores: a three hour review session, a one-day program, and an intensive 10-week course. The GMAT is usually taken by students from the colleges of Business, Engineering, and Arts and Sciences. For each of the test review strategies, two students from each college will be chosen at random. Perform a two-factor experiment on the GMAT scores given.

Solution:
Let factor A be the test review strategy and factor B be the specific college. The scores are shown in the table below.

		Factor B: College		
		Business	**Engineering**	**Arts and Sciences**
Factor A: Test Review Strategy	**3-hour review**	500	540	480
		580	460	400
	1-day program	460	560	420
		540	620	480
	10-week course	560	600	480
		600	580	410

Since two students from each college were chosen, then the number of replications, *r*, is 2. Also, each factor has three levels, so $a = b = 3$. Hence, the total number of observations, n_T, is 18. The next table includes all of the calculations needed to compute the Sum of Squares column.

		Factor B: College			**Row Totals**
		Business	**Engineering**	**Arts and Sciences**	
Factor A: Test Review Strategy	**3-hour review**	500 580 $\bar{x}_{11} = \frac{500+580}{2} = 540$	540 460 $\bar{x}_{12} = \frac{540+460}{2} = 500$	480 400 $\bar{x}_{13} = \frac{480+400}{2} = 440$	2960
		Factor A Mean: $\bar{x}_{1.} = \frac{2960}{6} \approx 493.33$			
	1-day program	460 540 $\bar{x}_{21} = \frac{460+540}{2} = 500$	560 620 $\bar{x}_{22} = \frac{560+620}{2} = 590$	420 480 $\bar{x}_{23} = \frac{420+480}{2} = 450$	3080
		Factor A Mean: $\bar{x}_{2.} = \frac{3080}{6} \approx 513.33$			
	10-week course	560 600 $\bar{x}_{31} = \frac{560+600}{2} = 580$	600 580 $\bar{x}_{32} = \frac{600+580}{2} = 590$	480 410 $\bar{x}_{33} = \frac{480+410}{2} = 445$	3230
		Factor A Mean: $\bar{x}_{3.} = \frac{3230}{6} \approx 538.33$			
	Column Totals	3240	3360	2670	**9270**
	Factor B Means	$\bar{x}_{.1} = \frac{3240}{6} = 540$	$\bar{x}_{.2} = \frac{3360}{6} = 560$	$\bar{x}_{.3} = \frac{2670}{6} = 445$	

The last bit of information, which could not be squeezed into the table, is the sample mean of all the 18 observations: $\bar{\bar{x}} = \frac{9270}{18} = 515$. The table above contains factor A (row) means, factor B (column) means, and means for each of the nine treatments. Now, we can compute the Sum of Squares column using the steps outlined above.

1. $\text{SST} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{2}\left(x_{ijk} - 515\right)^2 = (500-515)^2 + (580-515)^2 + (540-515)^2 + \ldots + (410-515)^2 = 82450$

2. $\text{SSA} = (3)(2)\sum_{i=1}^{3}\left(\bar{x}_{i.} - 515\right)^2 \approx 6[(493.33-515)^2 + (513.33-515)^2 + (538.33-515)^2] \approx 6100$

3. $\text{SSB} = (3)(2)\sum_{j=1}^{3}\left(\bar{x}_{.j} - 515\right)^2 = 6[(540-515)^2 + (560-515)^2 + (445-515)^2] = 45300$

4. $\text{SSAB} = 2\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\bar{x}_{ij} - \bar{x}_{i\cdot} - \bar{x}_{\cdot j} + 515\right)^2$ This one needs a little explanation first. Essentially, the expression inside the parentheses says to do this: take every treatment mean minus its factor A mean minus its factor B mean plus the mean of all observations. With this in mind, we get the following: $2[(540-493.33-540+515)^2 + (500-493.33-560+515)^2 + \ldots + (445-538.33-445+515)^2] \approx 11200$
5. $\text{SSE} = 82450 - 6100 - 45300 - 11200 = 19850$

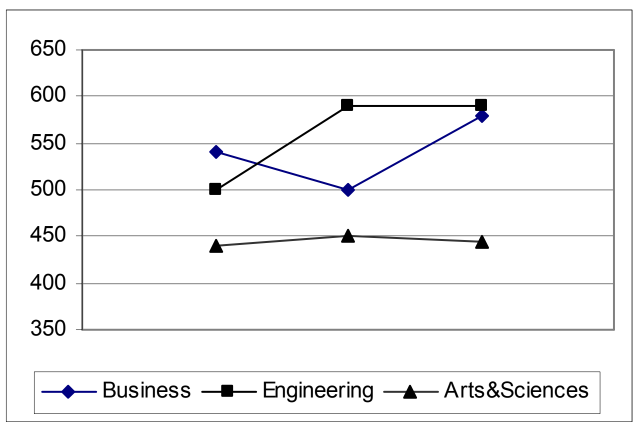
Using these values above, we can now finish the two-way ANOVA table.

Source of Variation	**Sum of Squares**	**Degrees of Freedom**	**Mean Square**	***F***	***p*-value**
Factor A	6100	2	3050	1.38	0.300
Factor B	45300	2	22650	10.27	0.005
Interaction	11200	4	2800	1.27	0.350
Error	19850	9	2206		
Total	82450	17			

For the *p*-value, use the following command on the TI-83/84:
2nd → DISTR → Fcdf(*F* value, 1EE99, df_1, df_2)

Let $\alpha = .05$. To find the *F*-critical value for factors A and B, df_1 equals the factor's degree of freedom and df_2 equals the degree of freedom of the error. In our case, $df_1 = 2$ and $df_2 = 9$. With this, we get the critical value to be 4.26. Since $1.38 < 4.26$, then we cannot conclude that there is any significant difference between the test review strategies. However, since $10.27 > 4.26$, then we can conclude that there is a difference in GMAT scores between the three colleges. To find out if there is any significant interaction effect, we need a new *F*-critical value. With $df_1 = 4$ and $df_2 = 9$, we get the critical value to be 3.63. Since $1.27 < 3.63$, then we cannot conclude that there is any significant interaction effect between the two factors.

Note: Although a graph of the mean test scores may suggest an interaction effect, especially with the Business and Engineering Colleges, in fact, the statistical analysis reveals no significant interaction. Trust the statistics!



Books critical to this handout:
1. "Statistics for Business and Economics 7ed." by Anderson, Sweeney, Williams
2. "Understandable Statistics" by Charles and Corrinne Brase