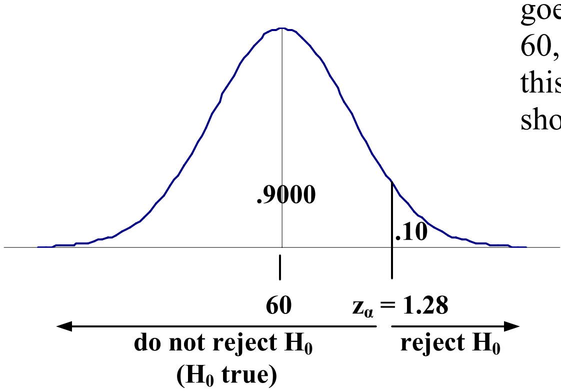
Calculating β

When you conduct an hypothesis test, α is the probability of making a Type I error (rejecting H_0 when it is actually true) and β is the probability of making a Type II error (failing to reject H_0 when it is actually false). We don't need to "calculate" α. It is the significance level of the test and is simply chosen before conducting the hypothesis test.

Consider the test:

H_0: $\mu \leq 60$
H_a: $\mu > 60$



Suppose you choose α = .10
Draw a normal curve with a mean of 60. Mark off an upper tail with an area of .10 (α) and look up the z value that corresponds to a tail area of .10 (called z_α or "critical z"). Your reasoning goes like this: "If H_0 is true with a population mean of 60, then the sampling distribution of $\bar{x}$ will look like this picture. So 90% of the time, my sample mean $\bar{x}$ should have a z-score that is less than 1.28. If my sample mean z-score is less than 1.28, I will not reject H_0. If my sample mean z-score is greater than 1.28, I will reject H_0. Because there is a 10% chance that I may get a sample mean z-score that is greater than 1.28 even when H_0 is true, my probability of making a Type I error will be .10."

How do we calculate β for this test? You can't do it directly from the information given. If H_0 is true with a mean of 60, then I know what the distribution of $\bar{x}$ looks like. If H_0 is false ($\mu > 60$), I don't know what the specific value of the real mean is and can't draw a picture! So what we can do is ask "what if" questions about possible values of the alternative mean μ_a, like:

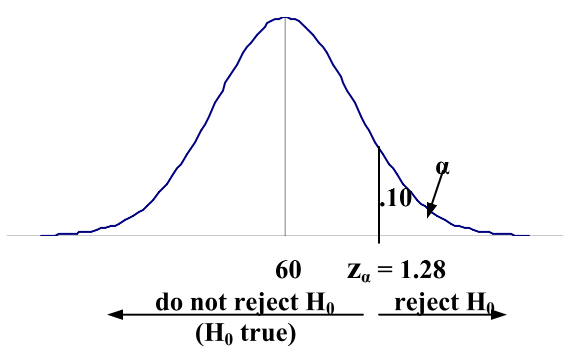
What is β if μ_a is really 62?
What is β if μ_a is really 65?
What is β if μ_a is really 70? etc.

In other words, β is not a set value like α, but instead **varies with μ_a**.

So let's ask the question "what is β if the mean is really 62?" This is the set up for our hypothesis test.

H_0: $\mu \leq 60$
H_a: $\mu > 60$
$\alpha = .10$

Suppose our sample size is 40 and our sample deviation is 12.



Start by changing z_α into an $\bar{x}$ value.

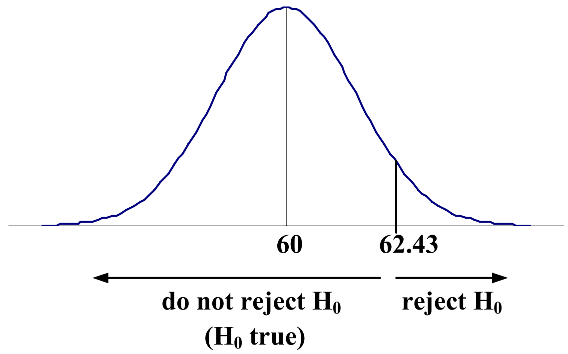
$$z = \frac{\bar{x} - \mu}{s / \sqrt{n}} \quad \rightarrow \quad 1.28 = \frac{\bar{x} - 60}{12 / \sqrt{40}}$$

$$1.28 = \frac{\bar{x} - 60}{1.897}$$

$$(1.28)(1.897) = \bar{x} - 60$$

$$2.43 = \bar{x} - 60$$

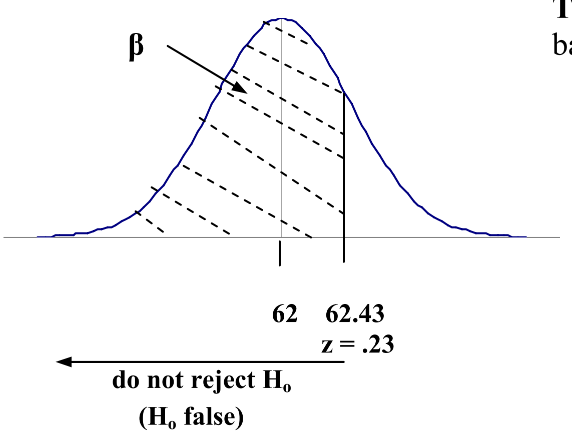
$$62.43 = \bar{x}$$



Next, state your DO NOT REJECT H_0 rule in terms of $\bar{x}$:

"I will not reject H_0 if my sample mean is less than 62.43."

Now, draw a new picture with a "what if" mean of 62 and locate 62.43 on this new distribution. You will not reject H_0 if your sample mean is <u>less than 62.43</u>. That would be the area to the left of 62.43 on the new "what if" distribution. **This area is β**. To find it , we need to turn 62.43 back into a z-score using the <u>new mean</u> of 62.



$$z = \frac{\bar{x} - \mu}{s / \sqrt{n}} = \frac{62.43 - 62}{12 / \sqrt{40}} = .23$$

Look up the area in the cumulative table for z = .23 to get **β = .5910**.

The probability of not rejecting H_0 when it is false, with a true mean of 62, is 59.1%.

- ❖ It is important that you draw these pictures! You need to "see" the area you are trying to calculate in order to use the tables correctly.
- ❖ In general, β gets larger as μ_a gets closer to the μ of your hypothesis test.
- ❖ The "power" of a test is 1 – β (the probability of NOT making a Type II error, or the probability of correctly rejecting H_0 when it is false).

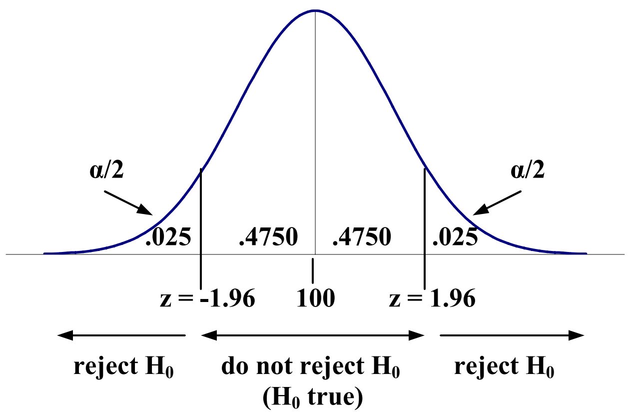
Two-tail example

Consider the hypothesis test:

H_0: $\mu = 100$
H_a: $\mu \neq 100$

$\alpha = .05$
$n = 50$
$s = 12$

Draw the picture for a 2-tailed test with $\mu = 100$.



Turn the critical z's into $\bar{x}$ values.

$$z = \frac{\bar{x} - \mu}{s/\sqrt{n}} \qquad -1.96 = \frac{\bar{x} - 100}{12/\sqrt{50}} \qquad 1.96 = \frac{\bar{x} - 100}{12/\sqrt{50}}$$

$$-1.96 = \frac{\bar{x} - 100}{1.7} \qquad 1.96 = \frac{\bar{x} - 100}{1.7}$$

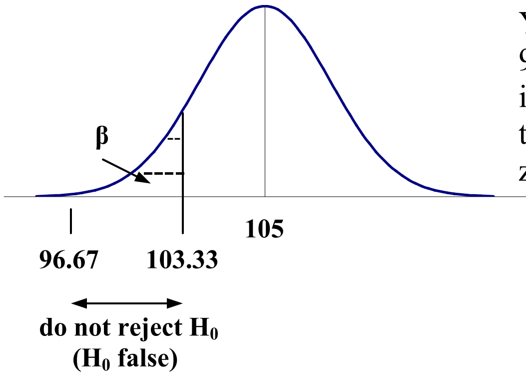
$$-3.33 = \bar{x} - 100 \qquad 3.33 = \bar{x} - 100$$

$$96.67 = \bar{x} \qquad 103.33 = \bar{x}$$

State your DO NOT REJECT rule in terms of the $\bar{x}$ values:

"I will not reject H_0 if my sample mean is between 96.97 and 103.33".

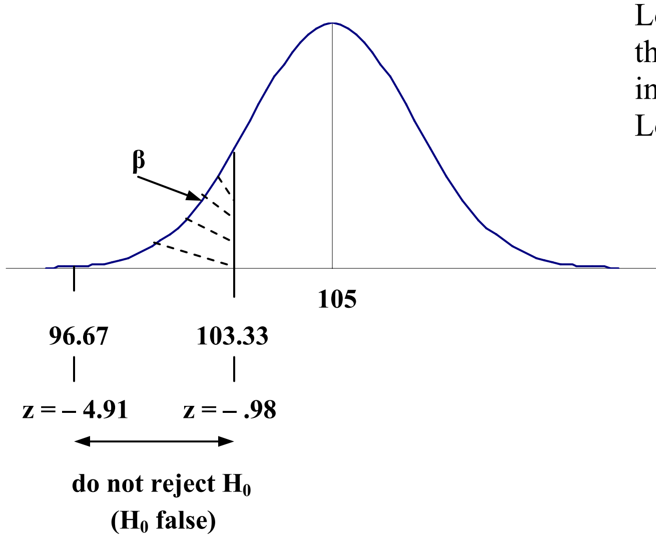
Of course, the probability α of making a Type I error is .05. What is the probability of making a Type II error if the mean is really 105? Draw a "what if" distribution with a mean of 105 and locate your $\bar{x}$ values on the new distribution.



You will not reject H_0 if your sample mean is between 96.67 and 103.33 (which will be a Type II error if the mean is really 105). Find the area between 96.67 and 103.33 on this new picture. **This area is β.** Turn the $\overline{x}$'s back into z's using the new mean of 105.

for 96.67: $$z = \frac{96.67 - 105}{12/\sqrt{50}} = -4.91$$

for 103.33: $$z = \frac{103.33 - 105}{12/\sqrt{50}} = -.98$$



Look up the tail area for z = - 4.91. It's off the charts! It's so far to the left that the area in the tail to the left is essentially 0.
Look up the tail area for z = - .98 to get

β = .1635.

The probability of not rejecting H_0 when it is false, with a true mean of 105, is 16.35%.

The probability of correctly rejecting H_0 when it is false with a true mean of 105 (not making a Type II error) is 1 – β, or 83.65% (the power of the test).