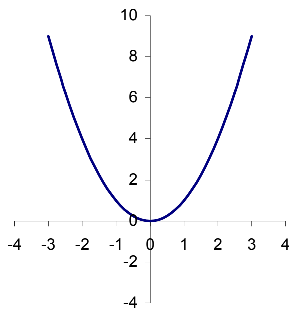
TRANSLATIONS OF GRAPHS, SYMMETRY, EVEN/ODD FUNCTIONS
(MAT 116)

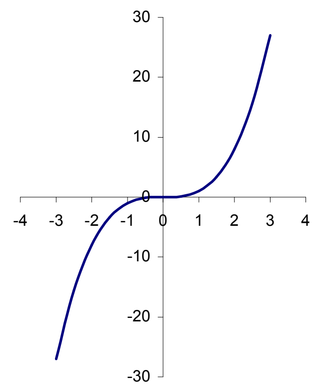
Given the graph of a basic function, you can graph variations on the basic function without having to actually calculate new y values by following a few simple rules for translations of the graph. A translation either moves the graph up/down or left/right; reflects the graph across a line; or stretches or shrinks it. Here's how it works.

Let's look at two basic functions:

$f(x) = x^2$



$f(x) = x^3$

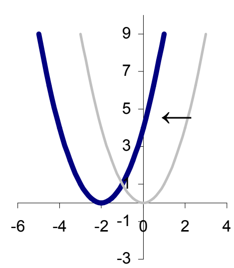


Horizontal Translation

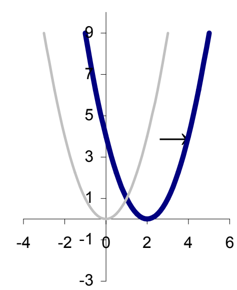
If you add to or subtract from the value of x "inside the function", that is, before any other operation is done to the x value, the graph shifts either left or right – horizontally.

$y = f(x + a)$ → graph moves **left** by a units
$y = f(x - a)$ → graph moves **right** by a units

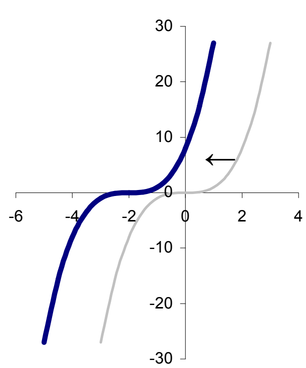
$y = (x + 2)^2$



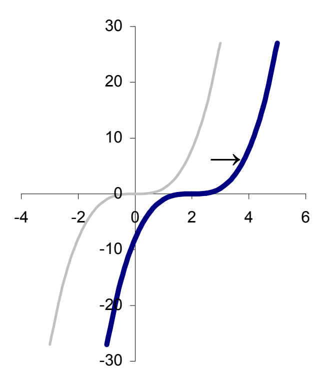
$y = (x - 2)^2$



$y = (x + 2)^3$



$y = (x – 2)^3$

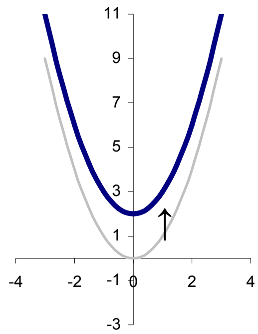


Vertical Translation

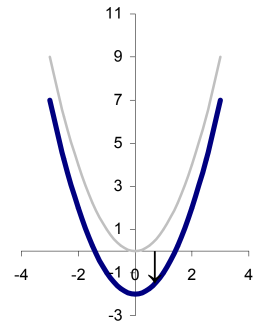
If you add to or subtract from the *function value* (the initial y value), the graph shifts either up or down – vertically.

y = f(x) + a → graph moves **up** by *a* units
y = f(x) – a → graph moves **down** by *a* units

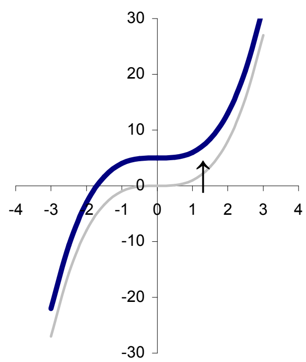
$y = x^2 + 2$
↑
original f(x)



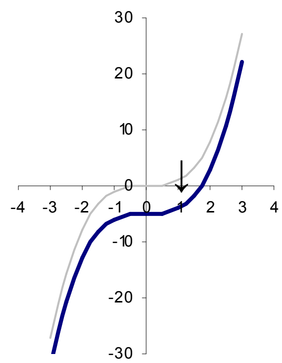
$y = x^2 – 2$



$y = x^3 + 5$
↑
original f(x)



$y = x^3 - 5$

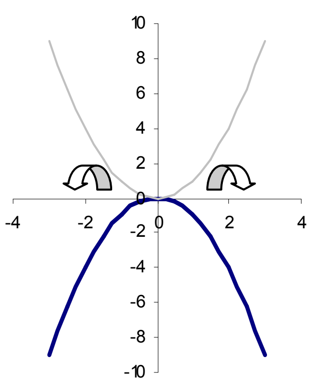


Reflection

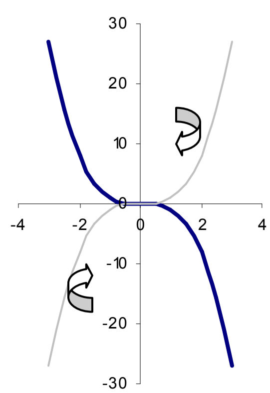
If the sign of either the x value or the function (y) value changes, the graph is reflected (as in mirror reflection).

y = – f(x) → graph reflects across the **x-axis**
y = f(–x) → graph reflects across the **y-axis**

$y = –(x^2)$



$y = (–x)^3$



Shrinking and Stretching

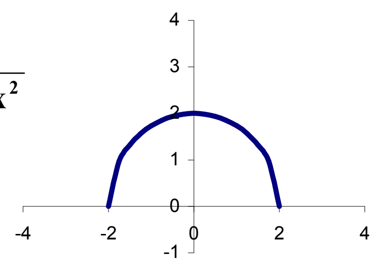
If the **function (y) value** is <u>multiplied</u> by a number other than 1 or 0, the graph will either *shrink* (compress) or *stretch* **vertically**. If the **x value** is <u>multiplied</u> by a number other than 1 or 0, the graph will either shrink or stretch **horizontally**. Here are the rules:

$$y = c \cdot f(x)\begin{cases} 0 < c < 1 \rightarrow \text{graph shrinks vertically} \\ c > 1 \rightarrow \text{graph stretches vertically} \end{cases}$$

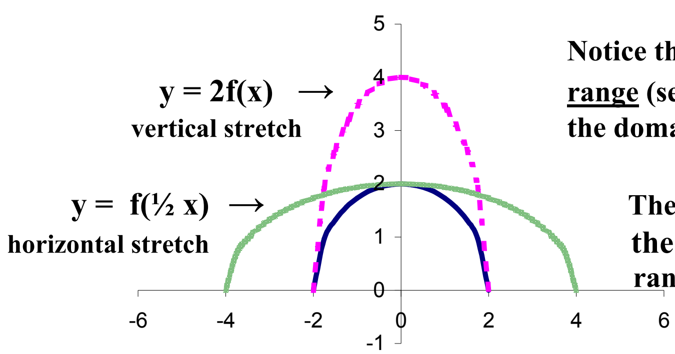
$$y = f(c \cdot x)\begin{cases} 0 < c < 1 \rightarrow \text{graph stretches horizontally} \\ c > 1 \rightarrow \text{graph shrinks horizontally} \end{cases}$$

With the graphs of some functions, it's hard to tell whether the graph is shrinking vertically or stretching horizontally (and vice-versa) when a multiplier is thrown in. So let's look at a graph where the distinction is clear.

$$f(x) = \sqrt{4 - x^2}$$



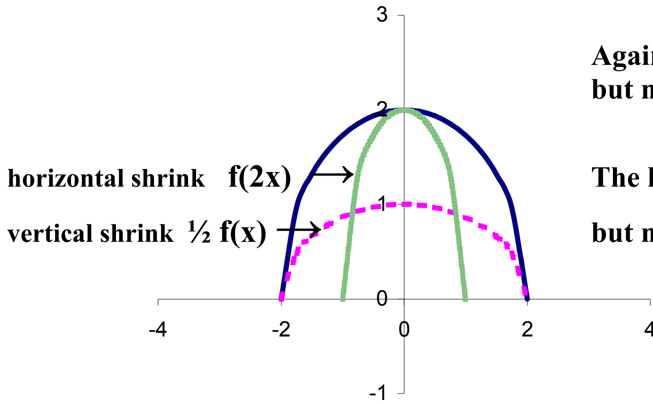
First, let's look at the difference between vertical stretch and horizontal stretch. We'll multiply **f(x)** by **2** – stretch it vertically – and, separately, multiply **x** by ½ - stretch it horizontally.



Notice that the vertical stretch changes the <u>range</u> (set of y-values), but does <u>not change</u> the domain (the x-values) of the function.

The horizontal stretch changes the domain of x, but does not change the range of y.

Now we'll multiply **f(x)** by ½ - shrink it vertically – and, separately, multiply **x** by **2** – shrink it horizontally.



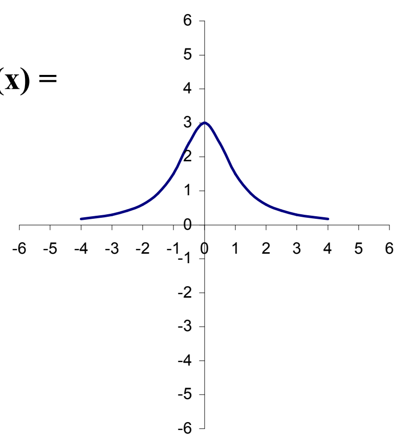
Again, the vertical shrink changes the range of y, but not the domain of x.

The horizontal shrink changes the domain of x,

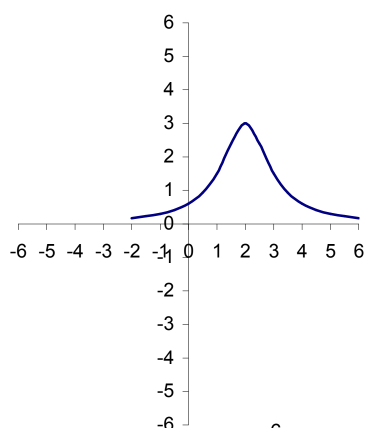
but not the range of y.

Let's walk through a graph translation without using the equation for the graph.

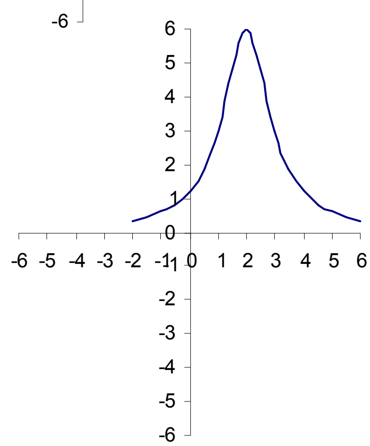
If f(x) =



what does – 2f(x-2) + 3 look like?

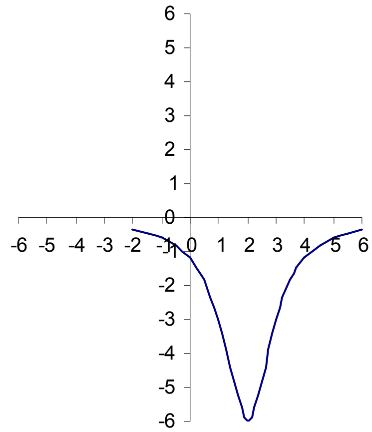


1) – 2f**(x-2)** + 3
↑
moves graph to the right by 2 units

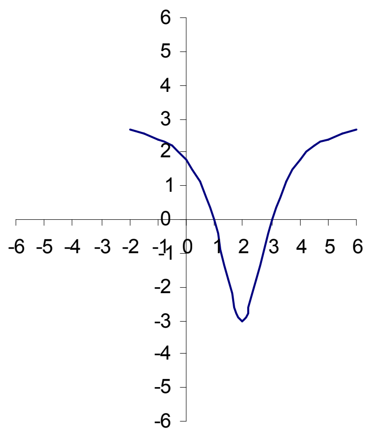


2) – **2f**(x-2) + 3
↑
stretches the graph vertically by a factor of 2

3) **–** 2f(x-2) + 3
↑
reflects the graph across the x-axis



4) – 2f(x-2) **+ 3**
↑
shifts the graph up 3 units



Done!

Symmetry

Recall the section on reflection of graphs:

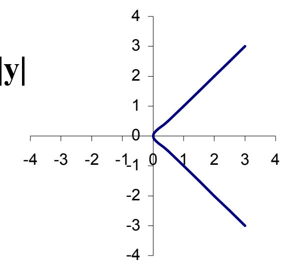
y = – f(x) → graph reflects across the **x-axis**
y = f(–x) → graph reflects across the **y-axis**

Now, if you do a reflection across an axis and the graph looks exactly the same as the original, it means that the graph is **symmetric** with respect to that axis.

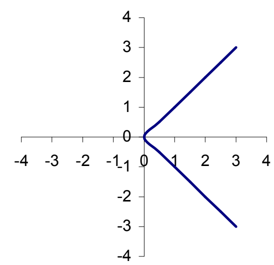
Concerning the equation for the graph, if a graph is symmetric about an axis:
either: replacing **y** with **–y** produces the same equation as the original (x-axis symmetry)
or: replacing **x** with **–x** produces the same equation as the original (y-axis symmetry)

Let's look at two graphs:

x = |y|

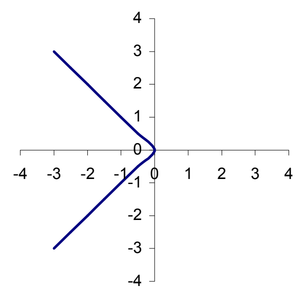


If we replace **y** with **–y,** the equation remains the same: **x = |-y| = |y|.** If we reflect the graph across the x-axis, it looks the same.

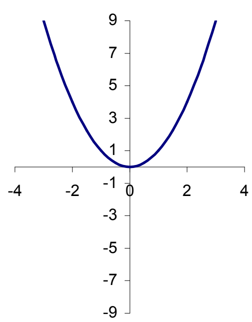


x-axis symmetry

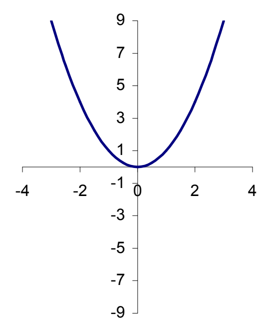
If we replace **x** by **–x,** the equation changes:
– x = |y| is not the same equation as **x = |y|.**
If we reflect the graph across the y-axis, it looks different.
It is not symmetric about the y-axis.



$y = x^2$

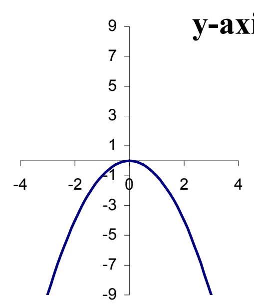


If we replace **x** with **–x** the equation remains the same: **$y = (-x)^2 = x^2$.** If we reflect the graph across the y-axis, it looks the same.



y-axis symmetry

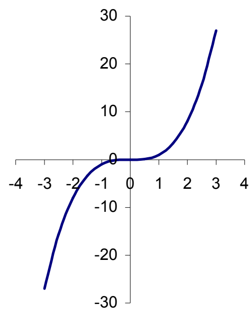
If we replace **y** by **–y**, the equation changes:
$-y = x^2$ is not the same equation as **$y = x^2$.**
If we reflect the graph across the x-axis, it looks different.
It is not symmetric about the x-axis.



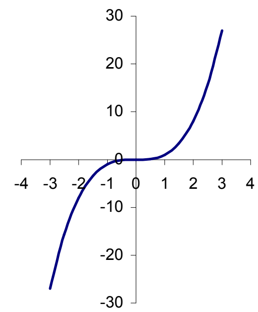
If we can replace <u>both</u> **y** by **–y** <u>and</u> **x** by **–x**, and the equation remains the same, the graph is symmetric with respect to the **origin** and looks the same as the original:

to test $\mathbf{y = x^3}$: $\mathbf{(-y) = (-x)^3} \rightarrow \mathbf{-y = -(x)^3} \rightarrow \mathbf{-y = -x^3}$
$\mathbf{-y = -x^3}$ is the <u>same</u> equation as $\mathbf{y = x^3}$

$\mathbf{y = x^3}$

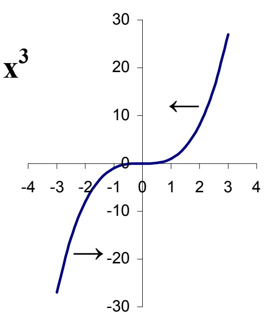


$\mathbf{-y = (-x)^3}$

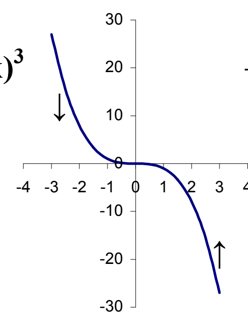


You can also think of symmetry about the origin as a <u>double reflection</u>. Replacing **x** with **–x** reflects the graph across the y-axis. Then, replacing **y** with **–y** in the new equation reflects the graph across the x-axis.

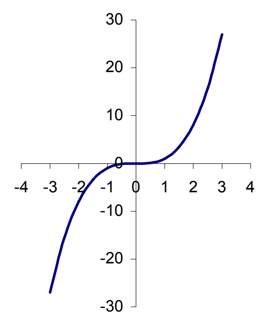
$y = \mathbf{x^3}$



→ $\mathbf{y = (-x)^3}$



→ $\mathbf{-y = (-x)^3}$



You will later encounter graphs that are symmetric about a line other than one of the axes or symmetric with respect to a point other than the origin.

Even/Odd Functions

One last idea concerning symmetry:

A function is called "**even**" if **$f(-x) = f(x)$**, such as our parabola **$y = x^2 = (-x)^2$**. This criterion also tells us that the graph is symmetric about the y-axis, or, if the graph is reflected across the y-axis, is looks the same as the original.

Note: Any function that has even powers of x (that is, of x by itself) – x^2, x^4, x^6, x^8, etc – and no x terms with odd powers, will be an even function.

Note 2: If the graph includes a horizontal translation, like **$(x + 2)^2$** or **$(x – 3)^4$**, it will not be an even function; replacing **x** by **–x** will change the **y** values. However, it can have a vertical translation, like **$x^2 + 3$,** and still be an even function.

A function is called "**odd**" if **$–f(x) = f(-x)$,** such as our cubic graph **$y = x^3$**, **$-y = (-x)^3$** (or the straight line equation **$y = x$**, or the equation **$xy = 6$**). By this definition, if a function is odd, it is symmetric about the origin, or, if the graph is reflected across both the x-axis and the y-axis, it looks the same as the original.

Note: Any function that has odd powers of x (that is, of x by itself) – x^1, x^3, x^5, x^7, x^9, etc – no x terms with even powers, and no vertical or horizontal translations will be an odd function.

□