Exponentials and Logarithms
Functions and Equations

Exponential and logarithm functions are very useful and widely used to model data in science, finance, and the social sciences. These functions are inverses of each other.

Exponential functions are of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. b is called the base and notice that the variable x is in the exponent position, in contrast to functions like x^n, where the variable x is the base being raised to an exponent. Exponential functions can be increasing ("exponential growth"), decreasing ("exponential decay"), or take on a variety of shapes with both increasing and decreasing sections.

Examples:

E.coli bacterial growth can be modeled by $N(t) = N_0(2)^{t/20}$, where N_0 is the initial number of bacteria and time t is measured in minutes. The bacteria population is increasing over time at an ever increasing rate.

The percentage of children in the U.S population can be modeled by $K(t) = 35.37(0.99)^t$, where t is the number of years since 1964. This is a slowly decreasing function.

The **logarithm function** is the inverse function of the exponential function.

If $b^y = x$ then $\log_b x = y$ ("log base *b* of x").
The log value gives you the power on the base *b* that produces *x* – **the log is an exponent**!

It's very important for both graphing and solving equations to be able to go back and forth between the exponential form and the logarithmic form – they express the same relationship.

Memorize: $\log_b x = y \quad \Leftrightarrow \quad b^y = x$

Change to exponential form: $\log_3 81 = 4$ (base: 3, exponent: 4)

Ans: $3^4 = 81$

Change to log form: $2^{-3} = \frac{1}{8}$ (base: 2, exponent: −3)

Ans: $\log_2(1/8) = -3$

The two most often used bases for logarithms are base 10 and base *e*. The "common logarithm" uses a base of 10 and is denoted as "**log**" on your calculator. "Log" with no base indicated is always base 10. The "natural logarithm" uses a base of *e* (an irrational number, approximately 2.718) and is denoted as "**ln**" on your calculator. "Ln" always has a base of *e*.

Change to exponential form: $\log 1000 = 3$

Ans: $10^3 = 1000$

Change to log form: $e^3 = x$

Ans: $\ln x = 3$

Basic properties: $\log_b b = 1$ (because $b^1 = b$) The log of it's own base is always 1.

$\log_b 1 = 0$ (because $b^0 = 1$) The log of 1 is always 0.

$\log_b\left(b^x\right) = x$ and $b^{\log_b x} = x$ These inverse functions "undo" each other.

Sometimes you can evaluate a logarithm just by changing it to exponential form.

Evaluate $\log_5 125$

Set up an equation: $\log_5 125 = x$

Change to exponential form: $5^x = 125$

125 is the cube of 5: $5^3 = 125$, so $x = 3$.

Evaluate $\log_{0.4} \frac{100}{16}$

Set up an equation: $\log_{0.4} \frac{100}{16} = x$

Change to exponential form: $0.4^x = \frac{100}{16}$

Rewrite 0.4: $\left(\frac{4}{10}\right)^x = \frac{100}{16}$

$\frac{100}{16}$ is the reciprocal of the square of $\frac{4}{10}$: $\left(\frac{4}{10}\right)^{-2} = \frac{100}{16}$, so $x = -2$

Evaluate $\ln e^5$

Using the property $\log_b\left(b^x\right) = x$ and knowing that *e* is the base of ln, we can go directly to the solution: $\ln e^5 = 5$.

More properties

The log of a product equals the sum of the logs of the factors of the product:

Product Rule: $\log_b(MN) = \log_b M + \log_b N$

The log of a quotient equals the difference of the logs of the terms in the quotient:

Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

An <u>exponent</u> inside a log can be brought out in front as a multiplier:

Power rule: $\log_b\left(M^p\right) = p\log_b M$

*** This property is very useful – it's the *only way* to get a variable out of an exponent position.

If two numbers are the same, then the logs are the same and vice versa (this may seem trivial, but it's important for solving equations!) This is the one-to-one property or the logarithm of both sides property.

$$M = N \quad \Leftrightarrow \quad \log_b M = \log_b N$$

<u>Expand</u> $\log_3 \dfrac{x\sqrt[3]{y}}{z^{(2/5)}}$ into logs of x, y, and z.

Using the quotient rule: $\log_3 \dfrac{x\sqrt[3]{y}}{z^{(2/5)}} = \log_3\left(x\sqrt[3]{y}\right) - \log_3 z^{(2/5)}$

Using the product rule, and changing to a rational exponent: $= \log_3 x + \log_3 y^{(1/3)} - \log_3 z^{(2/5)}$

Using the exponent rule: $= \log_3 x + (1/3)\log_3 y - (2/5)\log_3 z$

<u>Combine</u> the following logs into one log (with a coefficient of 1): $3\ln x - \ln\left(y\sqrt{z}\right) + 2\ln\left(\dfrac{x}{y}\right)$

If there are coefficients other than 1 in front of the logs, you must <u>always</u> <u>first</u> move those into the logs as exponents.

$$3\ln x - \ln\left(y\sqrt{z}\right) + 2\ln\left(\frac{x}{y}\right) = \ln x^3 - \ln\left(y\sqrt{z}\right) + \ln\left(\frac{x}{y}\right)^2$$

Using the quotient rule: $\ln x^3 - \ln\left(y\sqrt{z}\right) + \ln\left(\dfrac{x}{y}\right)^2 = \ln\dfrac{x^3}{y\sqrt{z}} + \ln\left(\dfrac{x}{y}\right)^2$

Using the product rule: $\ln\dfrac{x^3}{y\sqrt{z}} + \ln\left(\dfrac{x}{y}\right)^2 = \ln\left(\dfrac{x^3}{y\sqrt{z}}\left(\dfrac{x}{y}\right)^2\right) = \ln\left(\dfrac{x^5}{y^3\sqrt{z}}\right)$

Change of base

Since calculators usually have only two logarithm keys (*log* and *ln*), to evaluate a logarithm using a base different than either 10 or *e*, the following formula is used:

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$$

Note that you can use either the common log or the natural log – the answer's the same.

Example: $\log_{25} 73 = \dfrac{\log 73}{\log 25} = \dfrac{\ln 73}{\ln 25} \approx 1.33$

Evaluate without a calculator: $(\log_2 5)(\log_5 6)(\log_6 8)$

Do the changes of base and cancel: $\dfrac{\cancel{\log 5}}{\log 2} \bullet \dfrac{\cancel{\log 6}}{\cancel{\log 5}} \bullet \dfrac{\log 8}{\cancel{\log 6}} = \dfrac{\log 8}{\log 2} = \dfrac{\log 2^3}{\log 2} = \dfrac{3\cancel{\log 2}}{\cancel{\log 2}} = 3$

(Change of base can be used to change from any base to any other base: $\log_b x = \dfrac{\log_a x}{\log_a b}$.)

Domain and range of logarithmic functions

Logarithms cannot be computed for zero or negative numbers (why?). For the function $\log_b x$, the domain is all positive real numbers and the range is all real numbers.

Solving exponential and logarithmic equations

If a variable is "trapped" inside a logarithm or in the exponent position, the properties of logs and exponents can be used to "untrap" it and solve the equation.

Some simple exponential equations can be solved if there are exponentials with the same base on both sides: if $b^x = b^y$, then $x = y$.

Solve: $3^{4x-7} = 27$

Rewrite 27 as 3^3: $3^{4x-7} = 3^3$
Because the base is the same, the exponents are equal: $4x - 7 = 3$ and $x = 5/2$

If that can't be done, logarithms are used (and this is where that exponent rule comes in handy).

Solve: $7^x = 98$

Take the log of both sides: $\log 7^x = \log 98$
Bring out the exponent: $x \log 7 = \log 98$
Divide both sides by log 7: $x = (\log 98)/(\log 7) \approx 2.356$

Solve: $5^{2x+5} = 243$

Take the log of both sides: $\ln 5^{2x+5} = \ln 243$
Bring out the exponent: $(2x + 5)\ln 5 = \ln 243$
Divide both sides by ln 5: $2x + 5 = \dfrac{\ln 243}{\ln 5}$

Solve for x: $x = \frac{1}{2}\left(\frac{\ln 243}{\ln 5} - 5\right) \approx -0.793$

Notice that it doesn't matter if you use *log* or *ln*!

Logarithmic equations can often be solved by first converting them to their exponential form.

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$

<u>Solve</u>: $\log(3x - 14) = 2$

Remember the base of *log* is 10: $\log_{10}(3x - 14) = 2$

Convert to exponential form: $10^2 = 3x - 14 \rightarrow 3x - 14 = 100$

Solve for x: $3x = 114 \rightarrow x = 38$

<u>Solve</u>: $\log_3 x + \log_3 (x + 6) = 3$

Use the product rule to combine the logs: $\log_3(x(x+6)) = 3$

Convert to exponential form: $3^3 = x(x+6) \rightarrow 27 = x^2 + 6x$

Solve the quadratic: $x^2 + 6x - 27 = 0$

$(x + 9)(x - 3) = 0$

$x = 3$ (can't use x = -9 because it's not in the domain)

<u>Solve</u>: $\ln(1 - x) + \ln(3 - x) = \ln 8$

Use the product rule to combine the logs on the left: $\ln[(1 - x)(3 - x)] = \ln 8$

If the logs are equal, then the expressions within the logs are equal: $(1 - x)(3 - x) = 8$

$3 - 4x + x^2 = 8$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = -1$ (can't use x = 5 because it would make (1 – x) and (3 – x) negative)

<u>Solve</u>: $5^{2x} = 3^{x+2}$

Take the log of both sides (we'll use *ln* this time): $\ln 5^{2x} = \ln 3^{x+2}$

Bring out the exponents: $2x \ln 5 = (x + 2)\ln 3$

Clear the parentheses: $2x \ln 5 = x \ln 3 + 2\ln 3$

Group the x terms: $2x \ln 5 - x \ln 3 = 2\ln 3$

Factor out the x: $x(2\ln 5 - \ln 3) = 2\ln 3$

Solve for x: $x = \frac{2\ln 3}{2\ln 5 - \ln 3} = \frac{\ln 3^2}{\ln 5^2 - \ln 3} = \frac{\ln 9}{\ln(25/3)}$

(you can simplify the logs, but it's not necessary) ≈ 1.0363

Solve: $\ln x^{\ln x} = 4$

Bring out the exponent: $(\ln x)(\ln x) = 4 \rightarrow (\ln x)^2 = 4$

Take the square root: $\ln x = 2$ $\ln x = -2$

Solve by converting to exponential form: $e^2 = x$ $e^{-2} = x$

Check the answers:
Calculate the exponent of ln x: $\ln e^2 = 2 \ln e = 2$ $\ln e^{-2} = -2 \ln e = -2$
(Remember, a log of its own base is always 1.)

$$\ln(e^2)^{\ln(e^2)} = \ln(e^2)^2 = \ln e^4 = 4 \qquad \ln(e^{-2})^{\ln(e^{-2})} = \ln(e^{-2})^{-2} = \ln e^4 = 4$$

Solve: $(\log_3 x)^2 - \log_3 x^2 = 3$

This looks like a quadratic, except for that $\log_3 x^2$ term. But we can change that!

Using the exponent rule: $(\log_3 x)^2 - \log_3 x^2 = 3 \rightarrow (\log_3 x)^2 - 2 \log_3 x = 3$

Factor $(\log_3 x)^2 - 2 \log_3 x - 3 = 0$: $(\log_3 x - 3)(\log_3 x + 1) = 0$

Solve by converting to exponential form: $\log_3 x = 3$ $\log_3 x = -1$
$3^3 = x$ $3^{-1} = x$
$x = 27, 1/3$

Check the answers: $\log_3 27 = 3$ $\log_3 27^2 = 6$ $3^2 - 6 = 3$

$\log_3 (1/3) = -1$ $\log_3 (1/3)^2 = -2$ $(-1)^2 - (-2) = 3$

With equations involving similar but opposite sign exponentials, it makes things easier to first get rid of the negative exponent.

Solve: $e^x - 6e^{-x} = 1$

Multiply everything by e^x: $e^x(e^x - 6e^{-x} = 1) \rightarrow e^{2x} - 6e^0 = 1e^x$

This is now a quadratic: $e^{2x} - e^x - 6 = 0$

Factor: $(e^x - 3)(e^x + 2) = 0$

Solve for e^x: $e^x = 3$ $e^x = -2$

Solve for x: $\ln e^x = \ln 3$ $\ln e^x = \ln (-2)$
$x \ln e = \ln 3$ cannot take the log of a negative number!
$x = \ln 3$

Solve: $\dfrac{3^x + 3^{-x}}{3^x - 3^{-x}} = 8$

Multiply the left side by $\dfrac{3^x}{3^x}$: $\dfrac{3^x}{3^x}\left[\dfrac{3^x + 3^{-x}}{3^x - 3^{-x}}\right] = 8 \quad \rightarrow \quad \dfrac{3^{2x} + 3^0}{3^{2x} - 3^0} = 8 \quad \rightarrow \quad \dfrac{3^{2x} + 1}{3^{2x} - 1} = 8$

Clear the fraction: $3^{2x} + 1 = 8\left(3^{2x} - 1\right)$

Clear the parentheses: $3^{2x} + 1 = 8 \bullet 3^{2x} - 8$

Gather the x terms on one side: $9 = 7 \bullet 3^{2x}$

Divide by 7: $9/7 = 3^{2x}$

Take the log of both sides: $\ln(9/7) = \ln(3^{2x})$

Bring out the exponent: $\ln(9/7) = 2x \ln 3$

Solve for x: $x = \dfrac{\ln(9/7)}{2\ln 3} \approx 0.1144$

Multiple logs just involve multiple exponential conversions.

Solve: $\log(\log x) = 2$

Convert to exponential form: $10^2 = 100 = \log x$

Convert again: $10^{100} = x$

Here's a tough-looking one: Solve $\dfrac{2^x}{3^{1-x}} = 12^{x+1}$

Take the log of both sides: $\log\left(\dfrac{2^x}{3^{1-x}}\right) = \log 12^{x+1}$

Use the quotient rule on the left: $\log 2^x - \log 3^{1-x} = \log 12^{x+1}$

Bring out the exponents: $x \log 2 - (1-x)\log 3 = (x+1)\log 12$

Clear the parentheses: $x\log 2 - \log 3 + x\log 3 = x\log 12 + \log 12$

Gather the x terms on one side: $x\log 2 + x\log 3 - x\log 12 = \log 12 + \log 3$

Factor out the x: $x(\log 2 + \log 3 - \log 12) = \log 12 + \log 3$

You could simplify the logs: $x\log\left(\dfrac{2 \bullet 3}{12}\right) = \log(12 \bullet 3) \rightarrow x\log\left(\dfrac{1}{2}\right) = \log(36)$

Solve for x: $x = \dfrac{\log 36}{\log(1/2)} \approx -5.1699$

There's a cool and faster way to solve this by noting that the base 12 = (3)(4) and playing with the exponentials:

$$\frac{2^x}{3^{1-x}} = (3\bullet 4)^{x+1} = 3^{x+1}\bullet 4^{x+1}$$

$$2^x = 3^{1-x}\bullet 3^{x+1}\bullet 4^{x+1} = 3^2\bullet 4^{x+1} = 9\bullet\left(2^2\right)^{x+1} = 9\bullet 2^{2x+2}$$

$$2^x = 9\bullet 2^{2x+2}$$

$$\frac{2^x}{2^{2x+2}} = 9 \qquad \rightarrow \qquad 2^{-x-2} = 9$$

Now take the log of both sides: $\log 2^{-x-2} = \log 9$

Bring out the exponent: $(-x-2)\log 2 = \log 9$

Solve for x: $x = -\frac{\log 9}{\log 2} - 2 \quad \approx -5.1699$

Problems

Change to logarithmic form: 1) $4^3 = 64$ 2) $3^{-3} = \frac{1}{27}$ 3) $y = e^{2x}$

Change to exponential form: 4) $\log_2 \frac{1}{32} = -5$ 5) $\ln x = 4$ 6) $\log_8 x = -2$

Write the logarithms in terms of logs of x, y, and z: 7) $\log_2\left(\frac{\sqrt[3]{x}}{y^2}\right)$ 8) $\log_7\left(xy^4\sqrt{z}\right)$

Write the logs as a single log with a coefficient of 1. Simplify if possible:

9) $2\log_3 x - \frac{1}{4}\log_3 y + 5\log_3 z$ 10) $\ln(x^2 - y^2) - \ln(x + y)$

Solve: 11) $\log(x^2 - 3x) = 1$ 12) $\log_3 x + \log_3 (x - 8) = 2$ 13) $4^{2x} = 3^{(x-2)}$

14) $10^{\log(3x + 5)} = 9$ 15) $\ln(e^{3x}) = 12$ 16) $\frac{10^x + 10^{-x}}{2} = 8$

Answers

1) $\log_4 64 = 3$ 2) $\log_3 \frac{1}{27} = -3$ 3) $\ln y = 2x$ 4) $2^{-5} = \frac{1}{32}$

5) $e^4 = x$ 6) $8^{-2} = x$ 7) $\frac{1}{3}\log_2 x - 2\log_2 y$ 8) $\log_7 x + 4\log_7 y + \frac{1}{2}\log_7 z$

9) $\log_3\left(\frac{x^2 z^5}{\sqrt[4]{y}}\right)$ 10) $\ln\left(\frac{x^2 - y^2}{x + y}\right) = \ln(x - y)$ 11) $x = 5, -2$ 12) $x = 9$

13) $x = \frac{-2\ln 3}{2\ln 4 - \ln 3} \approx -1.313$ 14) $x = \frac{4}{3}$ 15) $x = 4$ 16) $x = \log(8 \pm 3\sqrt{7})$