Inverse Trigonometric Functions

Whereas trig functions operate on an angle and produce a number, *inverse* trig functions operate on a number and produce an angle.

For example: if $\sin 30^o = ½$, then the inverse function $\sin^{-1}(½) = 30^o$

The functions are written with what looks like a -1 exponent, but this notation <u>does not mean reciprocal.</u> When this exponent is attached to a <u>function</u> (not a number), it means an *inverse function*. Inverse trig functions are also sometimes written as arcsin (pronounced "arc sine", arccos ("arc cosine"), etc.

$\mathrm{Sin}^{-1}(½)$ can be read as "the angle whose sine is ½ ". In fact, inverse trig functions *are* angles:

$$\cos^{-1}(x) = \theta, \quad \tan^{-1}(1) = \pi/4, \quad \csc^{-1}(2) = 30^o.$$

To produce the inverse of any function, the original function must be a <u>one-to-one</u> function – one x value for each y value and one y value for each x value. If it is not, often the domain can be restricted in order to work with a section that is one-to-one. For example, to create the inverse of $f(x) = x^2$ (a parabola), the domain must be restricted to either $x \geq 0$ or $x \leq 0$. Otherwise, f(x) values will be repeated – *not* a one-to-one function.

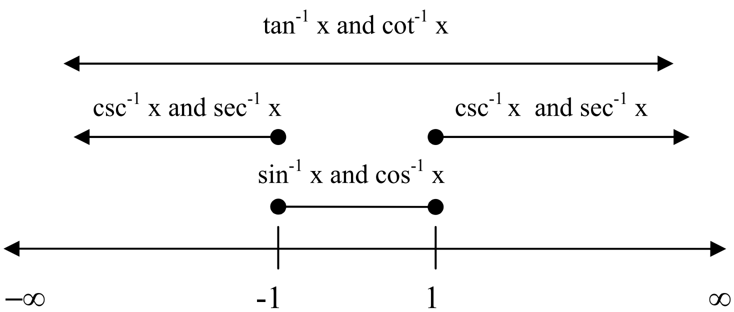
Since trig functions are circular (function values repeat), their domains must be restricted in creating their inverses, so as to maintain the one-to-one integrity. The restricted domains become the restricted ranges of the inverse functions. The ranges of the original trig functions do not change, and they become the domains of the inverse functions as is.

Range restrictions must be considered in solving inverse trig functions for the angle.
For example: $\sin^{-1} x = y$ *really means* "y is an angle in radians in the interval $[-\pi/2, \pi/2]$ whose sine is equal to x"; $\cot^{-1} x = y$ *really means* "y is an angle in radians in the interval $(0, \pi)$ whose cotangent is equal to x".

For calculator purposes, there are conversions for some of the inverse trig functions:

Function	=	Where x is in
$\sec^{-1} x$	$\cos^{-1}\left(\frac{1}{x}\right)$	$(-\infty, -1] \cup [1, \infty)$
$\csc^{-1} x$	$\sin^{-1}\left(\frac{1}{x}\right)$	$(-\infty, -1] \cup [1, \infty)$
$\cot^{-1} x$	$\tan^{-1}\left(\frac{1}{x}\right)$	$[0, \infty)$
$\cot^{-1} x$	$\tan^{-1}\left(\frac{1}{x}\right) + \pi$	$(-\infty, 0)$

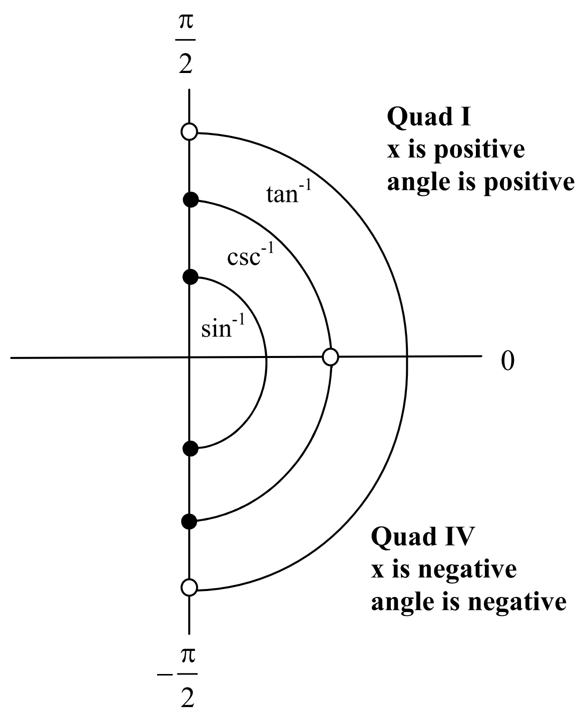
Domains of Inverse Trigonometric Functions:



Function	Domain
$\sin^{-1} x$	[-1, 1]
$\cos^{-1} x$	[-1, 1]
$\tan^{-1} x$	$(-\infty, \infty)$
$\cot^{-1} x$	$(-\infty, \infty)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$

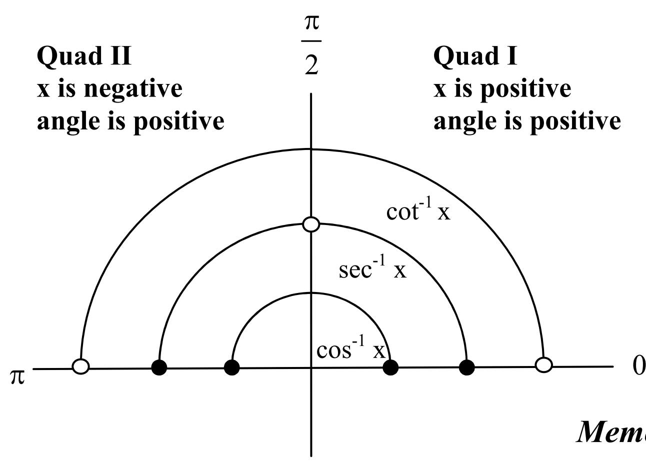
Ranges of Inverse Trigonometric Functions:

Function	Range
$\sin^{-1} x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\csc^{-1} x$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\tan^{-1} x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Memory aid:

sine related functions: **sin**, csc = $\frac{1}{\mathbf{sin}}$, tan = $\frac{\mathbf{sin}}{\cos}$



Function	Range
$\cos^{-1} x$	$[0, \pi]$
$\sec^{-1} x$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1} x$	$(0, \pi)$

Memory aid:

cosine related functions: **cos**, sec = $\frac{1}{\mathbf{cos}}$, cot = $\frac{\mathbf{cos}}{\sin}$

Finding Inverse Trig Function Values

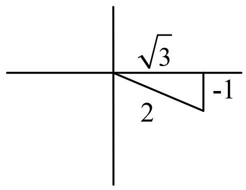
The Key (usually): **DRAW A PICTURE!**

1. Evaluate $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

a. What quadrants is $\tan^{-1} x$ in? QI and QIV. x is negative, so this angle is in QIV and the angle is negative.

b. Note that $-\frac{\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$, so we are looking for the angle whose tangent is $-\frac{1}{\sqrt{3}}$

c. Draw a picture:



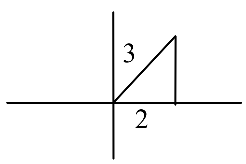
This is a familiar triangle: a 30°/60° triangle. The angle with a tangent of $-\frac{1}{\sqrt{3}}$ is -30°, or $-\frac{\pi}{6}$.

2. Evaluate $\sec^{-1}$ (3/2).

a. What quadrants is $\sec^{-1} x$ in? QI and QII. x is positive, so this angle is in QI and is positive.

b. If the secant of the angle is 3/2, then the cosine of the angle is $\frac{1}{\sec ant} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

c. Draw a picture:



d. Use your calculator and the conversion tip of $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$
(check the domain of x first!)

$$\cos^{-1}\left(\frac{2}{3}\right) \approx 48.2^{\circ} \text{ or } 0.84 \text{ radians}$$

3. Evaluate $\cot^{-1}$ (-.93170128).

a. What quadrants is $\cot^{-1} x$ in? QI and QII. x is negative, so this angle is in QII and is positive.

b. Use the calculator for this one: according to the conversion table, if x is in $(-\infty, 0)$, then $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right) + \pi$.

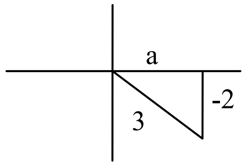
c. $\text{Tan}^{-1}\left(\frac{1}{-.93170128}\right) + \pi \approx -0.8207 + \pi \approx 2.32 \text{ rads or } 132.0^{\circ}$.

4. Find, without a calculator, $\cot\left(\arcsin\left(\frac{-2}{3}\right)\right)$.

a. Remember, the arcsin is an <u>angle</u>, so we are asked to find the cotangent of the angle <u>whose sin is -2/3</u>.

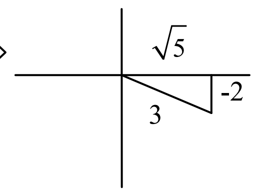
b. What quadrants is arcsin in? QI and QIV. x is negative, so this angle is in <u>QIV</u>.

c. Draw a picture:



d. Since we're looking for the cotangent, we need to find the third side.

$$a^2 + (-2)^2 = 3^2 \rightarrow a^2 = 9 - 4 = 5 \rightarrow a = \sqrt{5} \Rightarrow$$



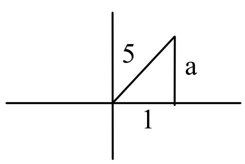
e. Cotangent $= \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{\sqrt{5}}{-2} = -\dfrac{\sqrt{5}}{2}$

5. Find, without a calculator, $\sin\left(2\cos^{-1}\frac{1}{5}\right)$

a. We are looking for the sine of <u>twice the angle</u> whose cosine is 1/5. This requires a double angle identity: sin 2A = 2sinAcosA. We know the cosine is 1/5, but we need to figure out the sine.

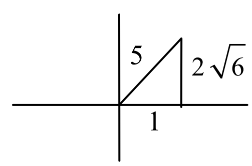
b. What quadrants is $\cos^{-1}$ in? QI and QII. x is positive, so the angle is in <u>QI</u>.

c. Draw a picture:



d. Find the missing side:

$$a^2 + 1^2 = 5^2 \rightarrow a^2 = 24 \rightarrow a = \sqrt{24} = 2\sqrt{6} \Rightarrow$$



e. The sine of the angle $= \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{2\sqrt{6}}{5}$.

f. Now work the identity: $\sin 2A = 2\sin A\cos A = 2\left(\dfrac{2\sqrt{6}}{5}\right)\left(\dfrac{1}{5}\right) = \dfrac{4\sqrt{6}}{25}$.