Graphing Exponential and Logarithmic Functions

Exponential functions play a large role in real life. From science to money, graphing these functions provide a visual representation to many applications.

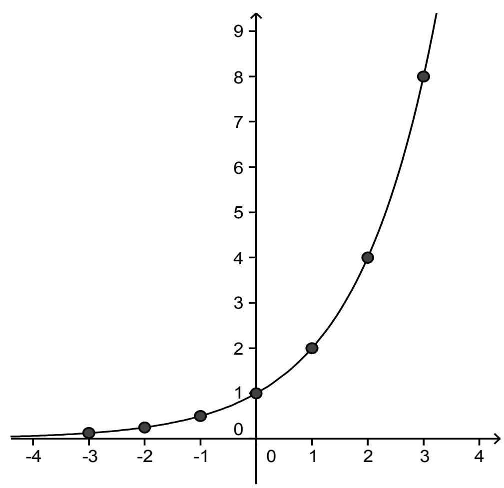
An exponential function is of the form b^x, where $b > 0$, $b \neq 1$, and *x* is any real number. For example, $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are exponential functions, but $h(x) = (-3)^x$ is not. The reason is that, for example, $(-3)^{1/2}$ gives a complex number, which cannot be graphed on the Cartesian plane. (Note that $f(x) = 2^x$ is not the same as $f(x) = x^2$! With an exponential function, the variable is <u>in the exponent, not the base</u>.)

Graphing exponential functions like $f(x) = 2^x$ simply requires evaluating *f*(*x*) with several real numbers to generate ordered pairs.

We will begin with discussing exponential functions, and then move on to logarithmic functions. Let's start with an example.

<u>Example 1</u>: Graph $f(x) = 2^x$.
Evaluate *f*(*x*) using the integers -3 to 3. This is by no means standard, but for a simple exponential function like this, -3 to 3 provides an adequate number of ordered pairs.

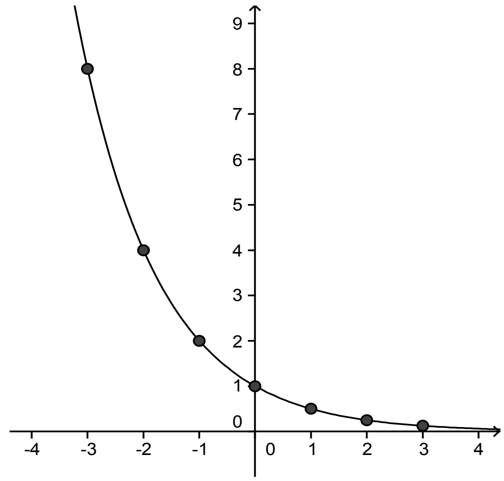
x	2^x	(*x*, *y*)
-3	$\frac{1}{8}$	(-3, 0.125)
-2	$\frac{1}{4}$	(-2, 0.25)
-1	$\frac{1}{2}$	(-1, 0.5)
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)



Connecting the points with a smooth curve completes the graph. Notice that as *x* approaches negative infinity, *f*(*x*) goes to zero, yet never crosses the *x*-axis.

<u>Example 2</u>: Graph $g(x) = \left(\frac{1}{2}\right)^x$.
Using the same integers from the above example, we get the following graph (next page):

x	$(1/2)^x$	(x, y)
-3	8	(-3, 8)
-2	4	(-2, 4)
-1	2	(-1, 2)
0	1	(0, 1)
1	$\frac{1}{2}$	(1, 0.5)
2	$\frac{1}{4}$	(2, 0.25)
3	$\frac{1}{8}$	(3, 0.125)



Notice this time that *g*(*x*) approaches zero as *x* approaches positive infinity, due to the base of *g*(*x*) being between zero and one. This leads to the following properties of exponential functions:

Properties of $f(x) = b^x$:

1. *f*(*x*) is a one-to-one function. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
2. The graph of *f*(*x*) is a smooth continuous curve passing through (1, *b*), with a *y*-intercept at (0, 1).
3. If $b > 1$, then *f*(*x*) is an increasing function that approaches zero as *x* approaches negative infinity.
4. If $0 < b < 1$, then *f*(*x*) is a decreasing function that approaches zero as *x* approaches positive infinity.

For properties (3) and (4), these imply that *f*(*x*) has a horizontal asymptote at the *x*-axis, with no *x*-intercept. However, this can change with translations.

Translations of Exponential Functions

Translations of exponential functions are similar to translations of other non-exponential functions, yet it can sometimes become tricky when different types of translations are used simultaneously.

Horizontal and Vertical

These two translations act the same way as for other functions. That is, if $f(x) = 2^x$, then $g(x) = 2^x - 3$ is shifted vertically down three units. Likewise, if $f(x) = 2^x$, then $g(x) = 2^{x-3}$ is shifted horizontally to the right three units.

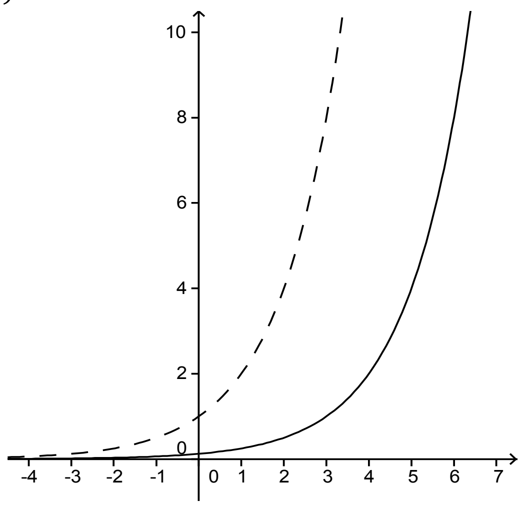
Reflection and Scaling

The shrinking/stretching and reflection translations also work in the usual way. Consider *g*(*x*) from Example 2. It can be rewritten as $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$, which is a reflection of 2^x across the *y*-axis.

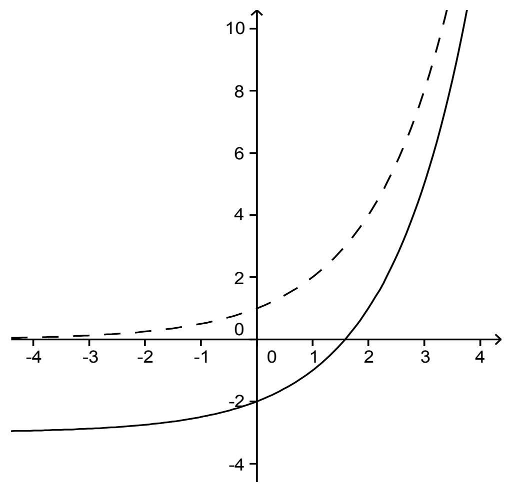
All of these translations are best demonstrated visually. With all of the following examples, the original function *f*(*x*) is graphed in a dashed style.

Translation Examples:

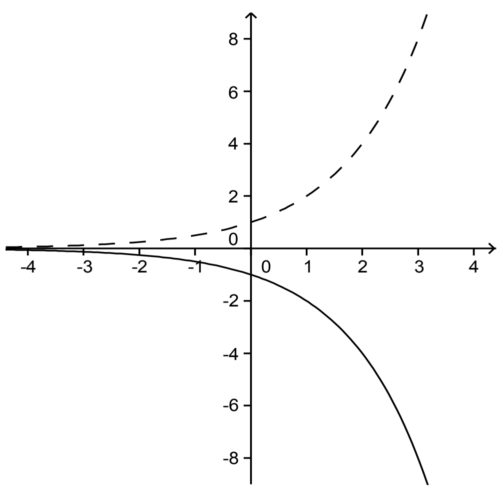
horizontal shift:
$f(x) = 2^x$
$g(x) = 2^{x-3}$



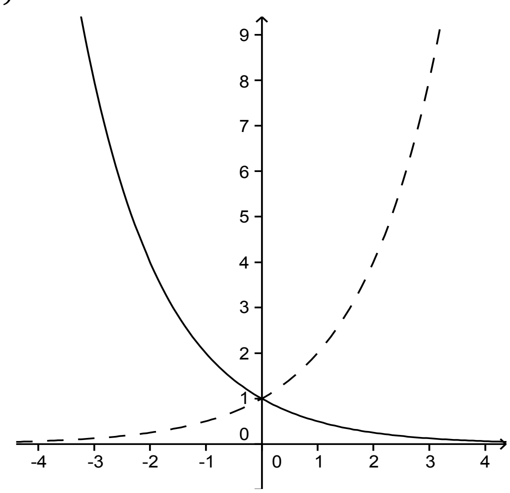
vertical shift:
$f(x) = 2^x$
$g(x) = 2^x - 3$



reflection across *x*-axis:
$f(x) = 2^x$
$g(x) = -2^x$



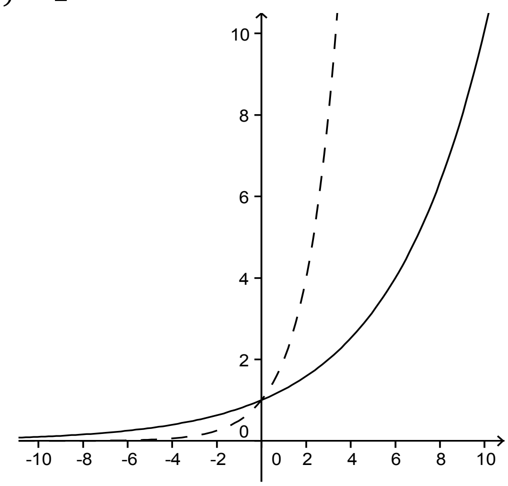
reflection across *y*-axis:
$f(x) = 2^x$
$g(x) = 2^{-x}$



horizontal stretch:

$f(x) = 2^x$

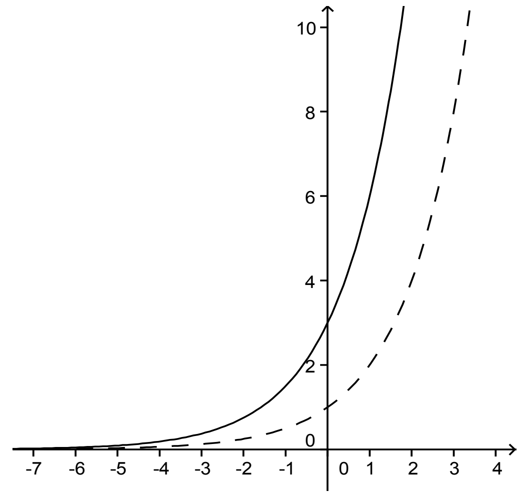
$g(x) = 2^{x/3}$



vertical stretch:

$f(x) = 2^x$

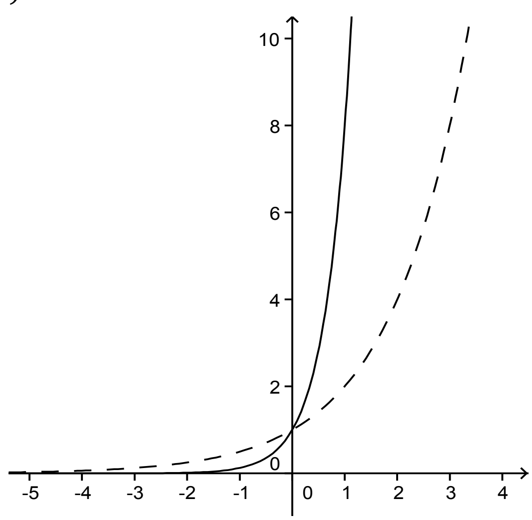
$g(x) = 3(2^x)$



horizontal shrink:

$f(x) = 2^x$

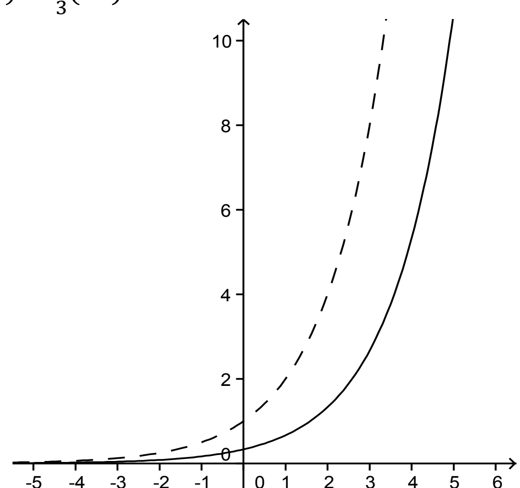
$g(x) = 2^{3x}$



vertical shrink:

$f(x) = 2^x$

$g(x) = \frac{1}{3}(2^x)$

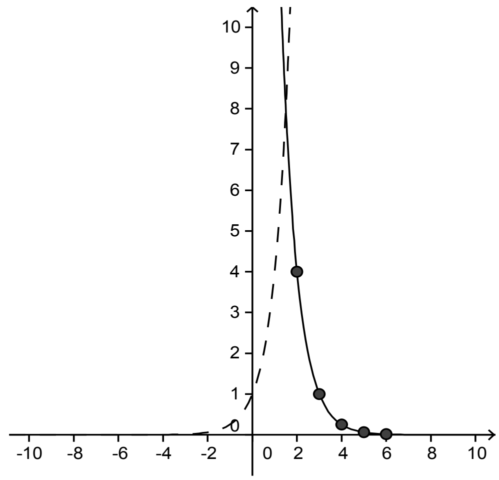


At first, the horizontal stretch/shrink may look like a horizontal shift. However, $f(x)$ and $g(x)$ have the same y-intercept, which would not happen with a horizontal shift. Likewise, a vertical stretch/shrink looks like a vertical shift. However, both $f(x)$ and $g(x)$ approach the x-axis, which would not happen with a vertical shift.

As mentioned earlier, individual horizontal and vertical translations operate the same as with other, non-exponential, functions. Yet strange results can occur when different translations are combined in an exponential function.

Example 3: Graph $f(x) = 4^{3-x}$.
Since 3 - *x* is zero when *x* = 3, we will choose *x* values around 3 in our table of values. Also, let's graph 4^x on the same axes for comparison.

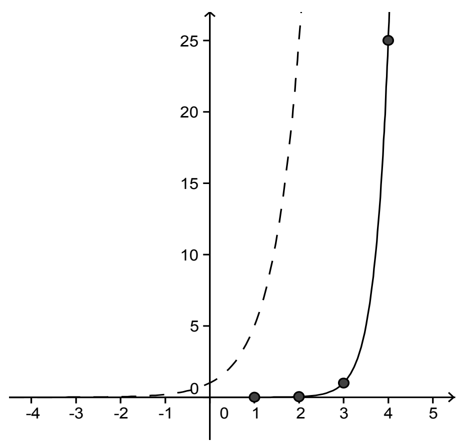
x	4^{3-x}	(*x*, *y*)
2	4	(2, 4)
3	1	(3, 1)
4	$\frac{1}{4}$	(4, 0.25)
5	$\frac{1}{16}$	(5, 0.0625)
6	$\frac{1}{64}$	(6, 0.01563)



At first, it looks like 4^{3-x} should reflect across the *y*-axis since *x* is negative. However, the graph tells a different story. Rewriting *f*(*x*), we get $4^{3-x} = 4^{-x+3} = 4^{-(x-3)}$. Therefore, *f*(*x*) actually shifts horizontally to the right 3 units, and then reflects across the vertical line *x* = 3.

Example 4: Graph $g(x) = 5^{2x-6}$.

x	5^{2x-6}	(*x*, *y*)
1	$\frac{1}{625}$	(1, 0.0016)
2	$\frac{1}{25}$	(2, 0.04)
3	1	(3, 1)
4	25	(4, 25)



This time it looks like *g*(*x*) should shift 6 units to the right. What does it really do? Rewriting, we get $5^{2x-6} = 5^{2(x-3)}$. So, *g*(*x*) actually shifts to the right three units, and then shrinks horizontally by a factor of two.

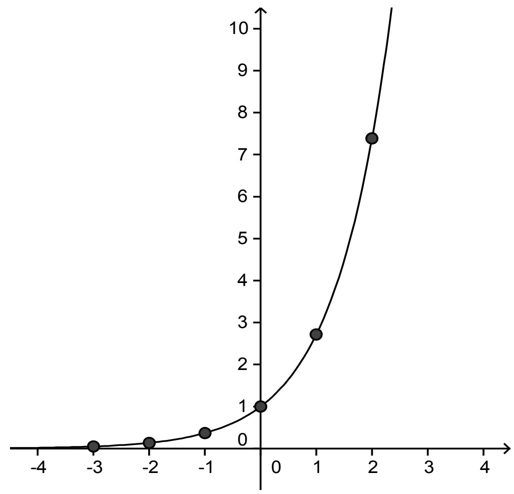
Exponential Functions with a Base of *e*

All of the above examples work when an exponential function has a base of *e*. That is, $f(x) = e^x$ is graphed in the same manner as $f(x) = 2^x$.

Example 5: Graph $f(x) = e^x$.

Use the same method as in Example 1. This time we will use integers -3 to 2 instead of -3 to 3, since $f(3) = e^3 \approx 20.1$ is too large for the graph in this example. (Use the "e^x" button on your calculator to approximate the *y*-values in the table below.)

x	e^x	(*x*, *y*)
-3	e^{-3}	(-3, 0.0498)
-2	e^{-2}	(-2, 0.1353)
-1	e^{-1}	(-1, 0.3679)
0	1	(0, 1)
1	e	(1, 2.7183)
2	e^2	(2, 7.3891)

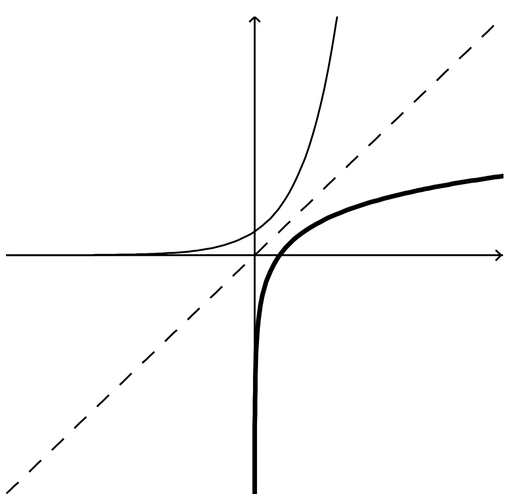


Note that just as with 2^x, e^x approaches zero as *x* approaches negative infinity.

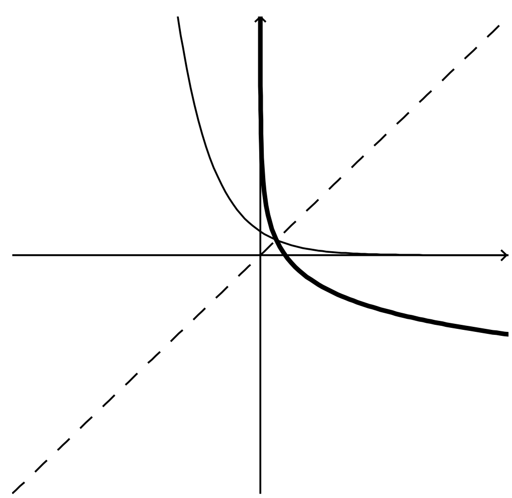
Graphing Logarithmic Functions

Note that earlier we mentioned that exponential functions are one-to-one. Then this means that an exponential function has an inverse function. We call these inverses *logarithm* or *logarithmic* functions. Consider the two graphs below (the dashed line is the function *y* = *x*).

(*b* > 1)
$f(x) = b^x$
$g(x) = \log_b x$ (in **bold**)
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$



(0 < *b* < 1)
$f(x) = b^x$
$g(x) = \log_b x$ (in **bold**)
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

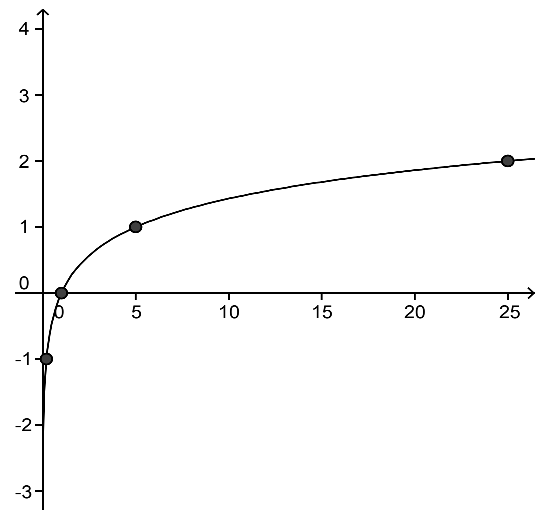


In each of these graphs the exponential and logarithmic functions are symmetric to each other across the line *y* = *x* (the second one may be a little hard to see at first). This verifies that exponential and logarithmic functions are inverses of each other. (Why?)

There are several ways to graph logarithmic functions. The easiest way to graph them by hand is to rewrite them in exponential form.

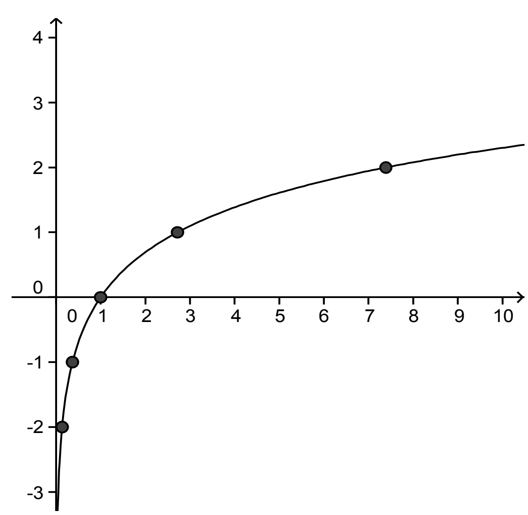
<u>Example 6</u>: Graph $f(x) = \log_5 x$.
Rewriting $f(x) = y = \log_5 x$ in exponential form we get $x = 5^y$. We can graph $x = 5^y$ by using the same method for exponential functions, except this time we will choose values for *y* and then compute the corresponding values for *x*.

$x = 5^y$	*y*	(*x*, *y*)
$\frac{1}{5}$	-1	(0.2, -1)
1	0	(1, 0)
5	1	(5, 1)
25	2	(25, 2)



<u>Example 7</u>: Graph $f(x) = \ln x$.
Since $\ln x = \log_e x$, the exponential form of *f*(*x*) is $x = e^y$. Choosing appropriate values for *y* and computing the values for *x*, we get our desired graph.

$x = e^y$	*y*	(*x*, *y*)
e^{-2}	-2	(0.1353, -2)
e^{-1}	-1	(0.3679, -1)
1	0	(1, 0)
e	1	(2.7183, 1)
e^2	2	(7.3891, 2)



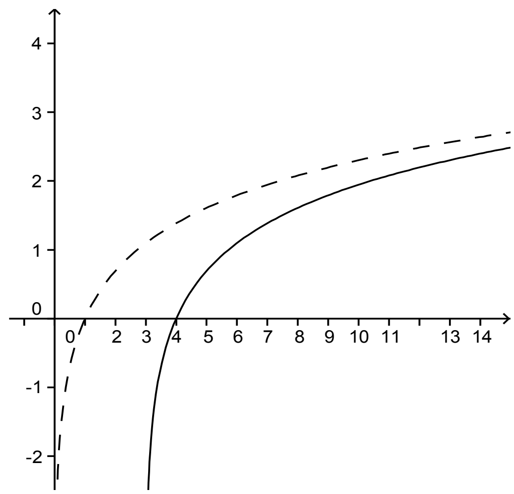
Translations of Logarithmic Functions:

We can perform translations of logarithmic functions like those that we did with exponential functions. As before, $f(x)$ is the dashed function.

horizontal shift:

$f(x) = \ln x$

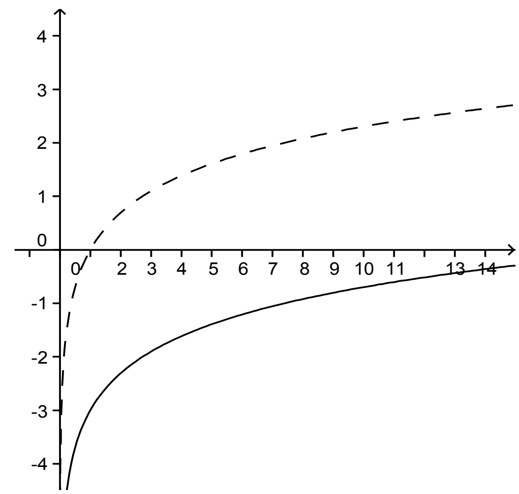
$g(x) = \ln(x-3)$



vertical shift:

$f(x) = \ln x$

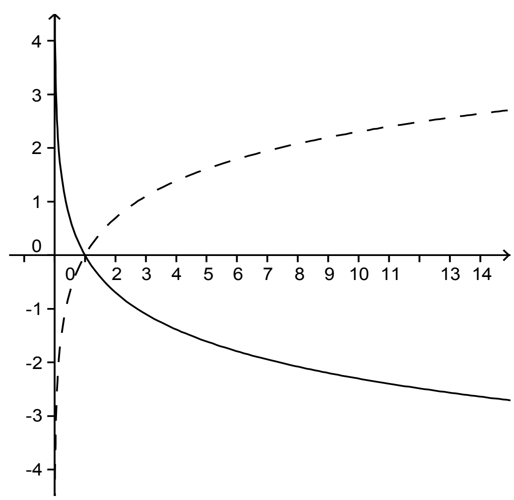
$g(x) = \ln x - 3$



reflection across *x*-axis:

$f(x) = \ln x$

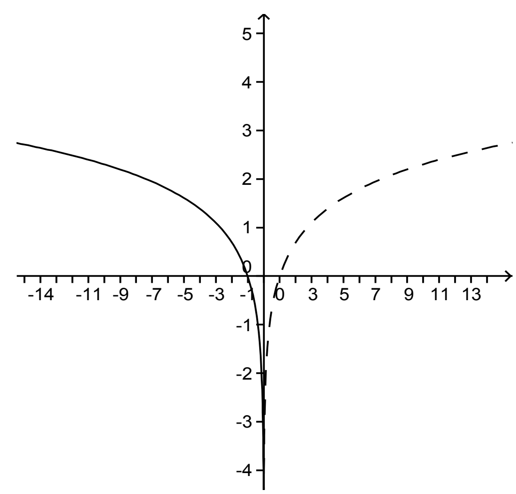
$g(x) = -\ln x$



reflection across *y*-axis:

$f(x) = \ln x$

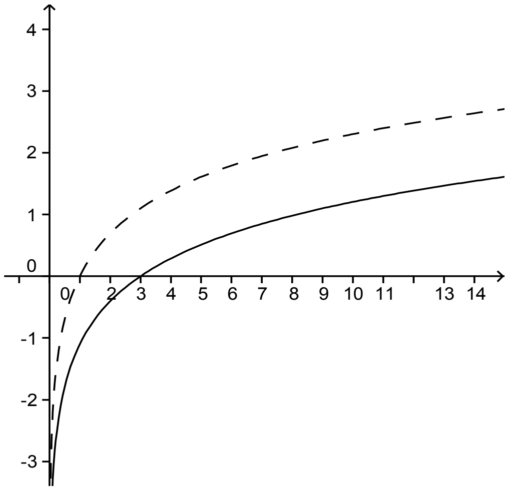
$g(x) = \ln(-x)$



horizontal stretch:

$f(x) = \ln x$

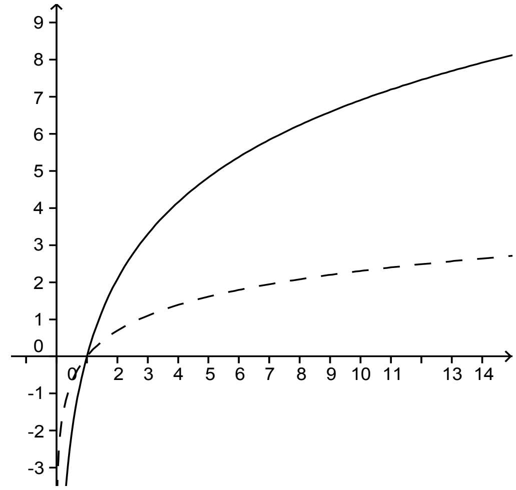
$g(x) = \ln(x/3)$



vertical stretch:

$f(x) = \ln x$

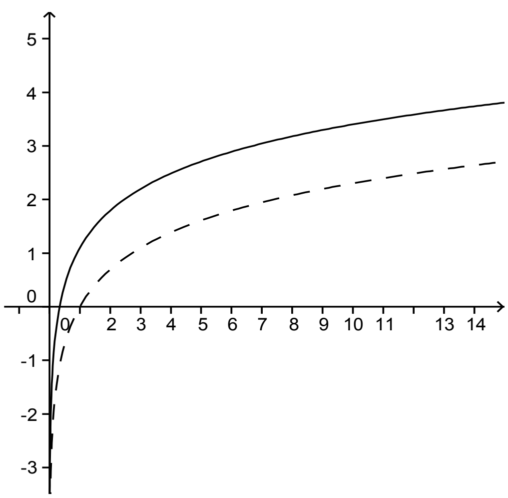
$g(x) = 3(\ln x)$



horizontal shrink:

$f(x) = \ln x$

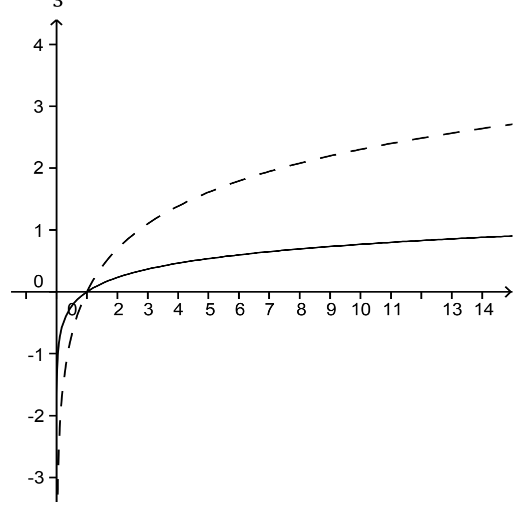
$g(x) = \ln(3x)$



vertical shrink:

$f(x) = \ln x$

$g(x) = \frac{1}{3}\ln x$



As with the transformations of exponential functions, there are patterns with the transformations of logarithmic functions. Do you see them?

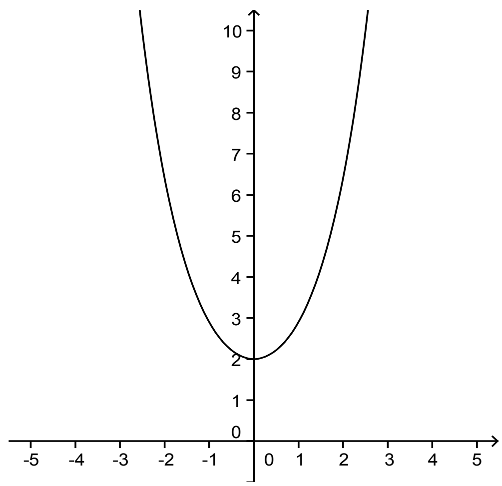
Using these transformations, we can describe how the graph of a logarithmic function can be obtained from a basic log function (similar to examples 3 and 4). For example, $g(x) = 3 - \frac{1}{2}\log x$ can be obtained from $f(x) = \log x$ by first shrinking *f*(*x*) by ½, then by reflecting it across the x-axis, and then finally shifting it up vertically by three units.

When Exponential and Logarithmic Functions “Break the Rules”

Sometimes you may encounter exponential and logarithmic functions whose graphs do not follow the usual patterns. Due to their complexity and uniqueness, these functions typically are graphed with a graphing calculator or computer software.

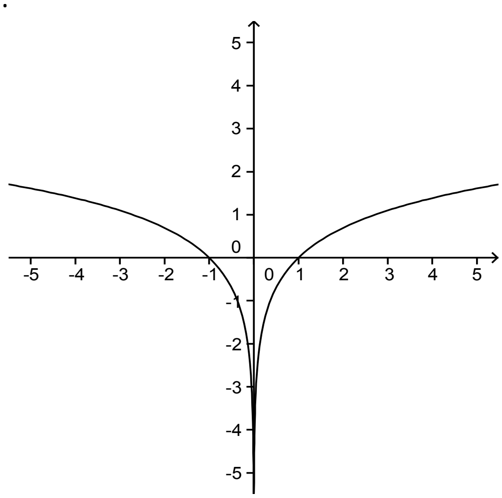
Example 8: Graph $h(x) = 2.5^x + 2.5^{-x}$.

Using a graphing calculator, we get the following graph:



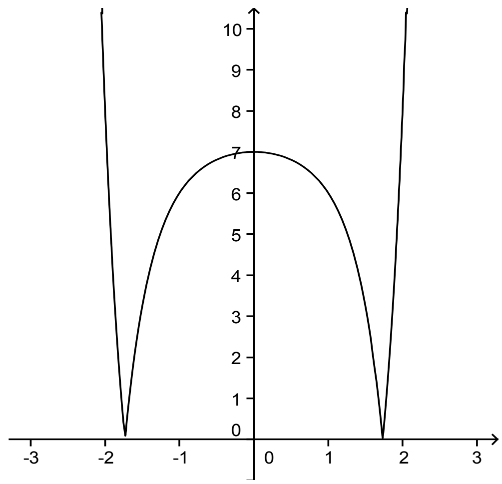
Notice the resemblance to $y = x^2 + 2$. It is not exactly the same, but it is very close.

Example 9: Graph $h(x) = \ln|x|$.



Normally, the domain of $\ln x$ is $(0, \infty)$. However, the domain of $\ln|x|$ is $(-\infty, 0) \cup (0, \infty)$. Why?

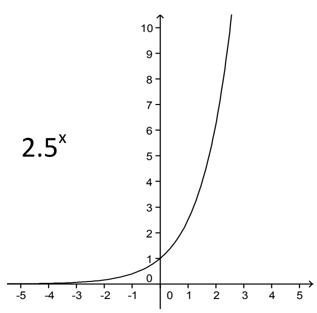
Example 10: Graph $h(x) = \left|2^{x^2} - 8\right|$.



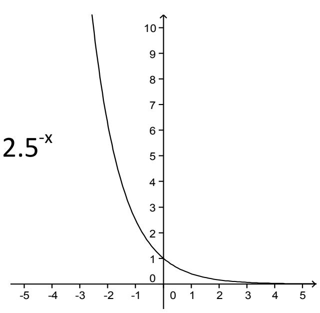
This graph does not fit the exponential pattern at all. Do you see the two culprits in the function?

If you have access to a graphing calculator or computer software, try experimenting with these "rogue" exponential and logarithmic functions (and have fun with it!). Try to figure out why you get the shapes that you do. For example, the function in Example 8 is an addition of 2.5^x and 2.5^{-x}. Compare these graphs to the graph in Example 8. Can you see what is happening?

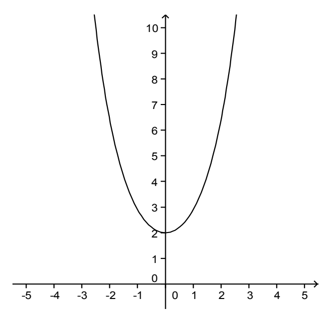
2.5^x



$+ \; 2.5^{-x}$



$=$



Problems

Graph the following equations:

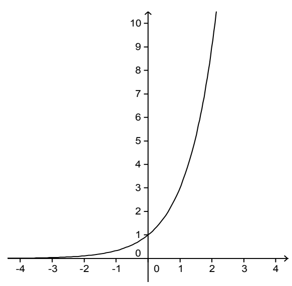
1.) $f(x) = 3^x$ 2.) $f(x) = \left(\frac{3}{4}\right)^x$ 3.) $f(x) = 3^{x+1}$ 4.) $f(x) = 4^x - 2$

5.) $f(x) = 5^{-x}$ 6.) $f(x) = -6^x$ 7.) $f(x) = e^{x/2}$ 8.) $f(x) = 3e^x$

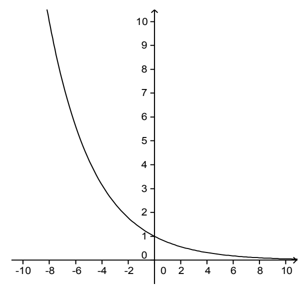
9.) $f(x) = 5^{2x}$ 10.) $f(x) = \frac{3^x}{6}$ 11.) $f(x) = 3^{x+2} - 4$ 12.) $f(x) = \log_6 x$

Answers

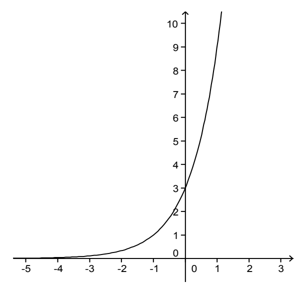
1.)



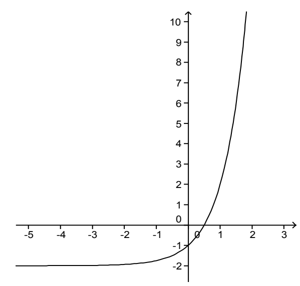
2.)



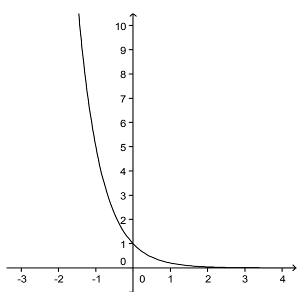
3.)



4.)



5.)



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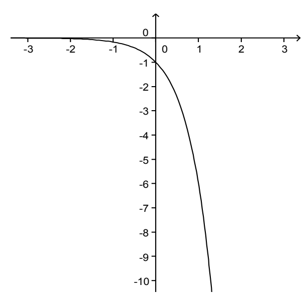

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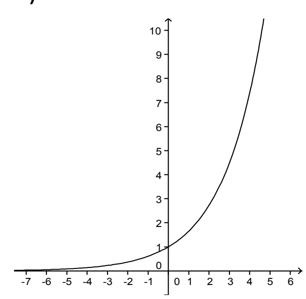
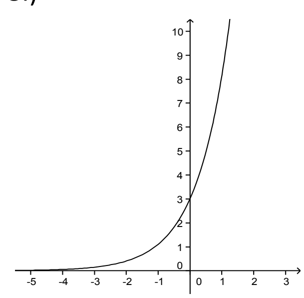
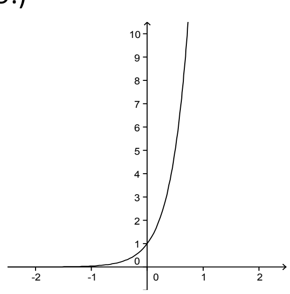


8.)



9.)



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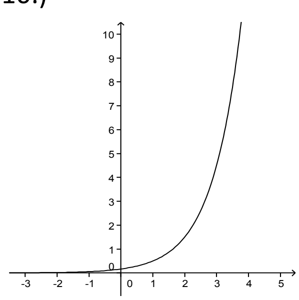
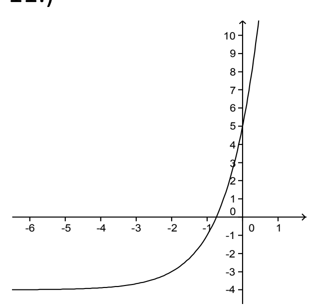


11.)



12.)