FINDING ZEROS OF POLYNOMIALS

It helps to have a systematic way of finding the zeros (or roots) of polynomials. The zeros are the x-intercepts, where $y = 0$. The zeros also give you the factors of the polynomial, and allow you to factor polynomials of higher degree than two (a quadratic). There are as many zeros as the degree of the polynomial, although the values may be repeated (multiplicity). Zeros of a polynomial may be rational or irrational, real or imaginary. Approaching the process systematically can save you a lot of time by restricting the possibilities.

We'll start with a relatively simple example.

Step 1. Use Descartes' Rule of Signs so you know what to expect.

Counting sign changes tells you how many positive and how many negative **real zeros** you might have.

A) For the polynomial $2x^3 \underbrace{+ x^2}_{} - \underbrace{25x +}_{} 12$ there are two sign changes as indicated.

This means there are at most two **real positive zeros**, but if the number you get is two or greater, subtract two until you can't subtract anymore – those numbers are also possible.

→ there are either **two or zero real positive zeros**

B) Substitute –x for x and repeat the process: $2(-x)^3 + (-x)^2 - 25(-x) + 12$

$-\underbrace{2x^3}_{} + x^2 + 25x + 12$ There is now one sign change. This means that at most there is **one negative real zero**.

So, the possibilities are:

Positive Real	**Negative Real**	**Imaginary (or complex)**
2	1	0
0	1	2

Note that we could have only an even number of complex zeros because the coefficients of the polynomial are real (you need conjugate pairs for that to happen).

Step 2. Use synthetic division to set upper and lower bounds on the real zeros.

If the coefficient on the leading term ($2x^3$ in this case) is <u>positive</u>, step up through the positive integers and divide until you get <u>all positive numbers</u> in the bottom row. This is the upper bound. If you get a zero remainder in the process, you've discovered a zero!

If the leading coefficient is <u>negative</u>, step up through the positive integers and divide until you get <u>all negative numbers</u> in the bottom row – this is the upper bound.

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -25 & 12 \\ & & 2 & 3 & -22 \\ \hline & 2 & 3 & -22 & -10 \end{array} \qquad \begin{array}{r|rrrr} 2 & 2 & 1 & -25 & 12 \\ & & 4 & 10 & -30 \\ \hline & 2 & 5 & -15 & -18 \end{array}$$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \underset{\textbf{zero}}{\underset{\uparrow}{0}} \end{array} \qquad \begin{array}{r|rrrr} 4 & 2 & 1 & -25 & 12 \\ & & 8 & 36 & 44 \\ \hline & 2 & 9 & 11 & 56 \end{array}$$

all positive → 4 is an upper bound

Now, step down through the negative integers and divide until you get <u>alternating signs</u> in the bottom row. This is the lower bound.

$$\begin{array}{r|rrrr} -1 & 2 & 1 & -25 & 12 \\ & & -2 & 1 & 24 \\ \hline & 2 & -1 & -24 & 36 \end{array} \qquad \begin{array}{r|rrrr} -2 & 2 & 1 & -25 & 12 \\ & & -4 & 6 & 38 \\ \hline & 2 & -3 & -19 & 50 \end{array}$$

$$\begin{array}{r|rrrr} -3 & 2 & 1 & -25 & 12 \\ & & -6 & 15 & 30 \\ \hline & 2 & -5 & -10 & 42 \end{array} \qquad \begin{array}{r|rrrr} -4 & 2 & 1 & -25 & 12 \\ & & -8 & 28 & -12 \\ \hline & 2 & -7 & 3 & \underset{\textbf{zero}}{\underset{\uparrow}{0}} \end{array}$$

also, alternating signs → -4 is a lower bound

Note that zero can be considered either positive or negative.

We've already found two zeros: $x = 3$ and $x = -4$. Since the Rule of Signs gave us only one negative, there should be another positive root. There are no complex roots because they would have to come in conjugate pairs to produce real coefficients and we already have two of the three zeros.

Step 3. Use the Rational Zero Theorem

If a polynomial has rational zeros, they must be in the set of possibilities given by this theorem. Use the leading coefficient and the ending constant term.

For $2x^3 + x^2 - 25x + 12$, the leading coefficient is 2 and the ending constant term is 12.

p is a factor of the constant 12 = ±1, ±2, ±3, ±4, ±6, ±12.
q is a factor of leading coefficient 2 = ±1, ±2

Find all possible combinations of $\frac{p}{q}$ = ±1, ±2, ±3, ±4, ±6, ±12, ±1/2, ±3/2

Test these values with synthetic division. But wait! Our upper and lower bounds exclude ±6 and ±12; we need another positive root; we've already tested 1, 2, and 4; so the only ones left to check are 1/2 and 3/2.

$$\begin{array}{r|rrrr} 1/2 & 2 & 1 & -25 & 12 \\ & & 1 & 1 & -12 \\ \hline & 2 & 2 & -24 & 0 \end{array} \quad \leftarrow \text{ our third } \textbf{zero} \text{ is } x = 1/2$$

Finding the zeros also leads you to the factors of the polynomial. Since our zeros are -4, ½, and 3, and the leading coefficient is 2, the polynomial $2x^3 + x^2 - 25x + 12$ factors into $(x + 4)(2x - 1)(x - 3)$.

What if none of the possible rational values work? Then there may be irrational or complex roots, and you would have to try another approach.

BIG SHORTCUT: Each time you find a zero, the polynomial is reduced by one degree. You can keep going with a new synthetic division from the bottom line of the previous division. Once you get down to a quadratic, you can solve it by factoring, square roots, or the quadratic formula.

For example: Once we found the first zero of the previous polynomial,

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & 0 \end{array}$$

because we started with a third-degree polynomial, the bottom line now corresponds to $2x^2 + 7x - 4$, which can be factored to $(2x - 1)(x + 4)$.

Find all solutions to: $4x^3 + 3x^2 + 16x + 12 = 0$.

1. Descartes' Rule of Signs tells us that there are no positive real zeros and there are either three or one negative real zeros. **The upper bound is zero**.

2. Find the lower bound for reals:

$$\begin{array}{r|rrrr} -1 & 4 & 3 & 16 & 12 \\ & & -4 & 1 & -17 \\ \hline & 4 & -1 & 17 & -5 \end{array} \leftarrow \textbf{alternating signs}$$

-1 is the lower bound

3. Find the rational zeros: Any rational zeros must be between 0 and -1. The possible $\frac{p}{q}$ values are -1/4, -1/2, -3/4, and -1. Testing shows that -3/4 works.

$$\begin{array}{r|rrrr} -3/4 & 4 & 3 & 16 & 12 \\ & & -3 & 0 & -12 \\ \hline & 4 & 0 & 16 & 0 \end{array}$$

4. The equation is now reduced to the quadratic $4x^2 + 16 = 0$.

$4x^2 = -16 \quad \rightarrow \quad x^2 = -4 \quad \rightarrow \quad x = 2i, x = -2i$

The three zeros are -3/4 and ±2*i*.

Imaginary zeros will not show up in the normal *x-y* coordinate system on a graphing calculator, so if you plot it, you'll only see the real zero at x = -3/4.

Find all solutions to: $x^4 - 3x^3 - 20x^2 - 24x - 8 = 0$

1. Descartes' Rule of Signs tells us that there could be one positive real zero and either three or one negative real zeros. Since there could be more negative real zeros than positive, let's start with those.

2. Find the lower bound:

$$\begin{array}{r|rrrrr} -1 & 1 & -3 & -20 & -24 & -8 \\ & & -1 & 4 & 16 & 8 \\ \hline & 1 & -4 & -16 & -8 & 0 \\ -2 & & -2 & 12 & 8 & \\ \hline & 1 & -6 & -4 & 0 & \end{array}$$

-1 is not a lower bound, but it is a zero.
The bottom line is translated: $x^3 - 4x^2 - 16x - 8$.

So let's continue from here, again looking for a lower bound.

Another zero! Now the polynomial is reduced to the quadratic $x^2 - 6x - 4 = 0$.
Since this isn't factorable, we'll use the quadratic formula.

$$x = \frac{6 \pm \sqrt{6^2 - 4(1)(-4)}}{2} = \frac{6 \pm \sqrt{52}}{2} = \frac{6 \pm 2\sqrt{13}}{2} = 3 \pm \sqrt{13}$$

The four zeros are -2, -1, and $3 \pm \sqrt{13}$

Find all solutions to: $x^4 + 2x^3 + 2x^2 - 4x - 8 = 0$

1. Descartes' Rule of Signs tells us there could be one positive real zero and either three or one negative real zeros.

2. Find the upper bound for real zeros.

$$1 \; \begin{array}{|rrrrr} 1 & 2 & 2 & -4 & -8 \\ & 1 & 3 & 5 & 1 \\ \hline 1 & 3 & 5 & 1 & -7 \end{array} \qquad 2 \; \begin{array}{|rrrrr} 1 & 2 & 2 & -4 & -8 \\ & 2 & 8 & 20 & 32 \\ \hline 1 & 4 & 10 & 16 & 24 \end{array}$$

all positive
2 is the upper bound

Find the lower bound for real zeros.

$$-1 \; \begin{array}{|rrrrr} 1 & 2 & 2 & -4 & -8 \\ & -1 & -1 & -1 & 5 \\ \hline 1 & 1 & 1 & -5 & -3 \end{array} \qquad -2 \; \begin{array}{|rrrrr} 1 & 2 & 2 & -4 & -8 \\ & -2 & 0 & -4 & 16 \\ \hline 1 & 0 & 2 & -8 & 8 \end{array}$$

alternating signs

We can consider zero to be negative, for looking at sign changes, so **-2 is the lower bound**.

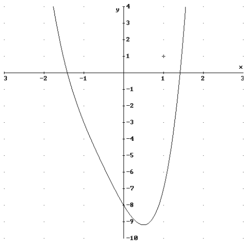
3. Find the rational zeros. All possible $\frac{p}{q}$ values are ±1, ±2, ±4, ±8. However, any rational zero must be between -2 and 2. So feasible $\frac{p}{q}$ values are ±1 and ±2. But we've already tested those values and they don't work! This means **there are no rational zeros**.

4. Possible zeros are 4 irrational, 2 irrational and 2 complex, or 4 complex.

To explore the roots:

1) Graph the polynomial on a graphing calculator. It appears there are two real roots. Using the calculator zero function produces values of $-\sqrt{2}$ and $\sqrt{2}$ (-1.4142, 1.4142).

2) Use math software or an advanced graphing calculator to either factor or find the zeros:



factors: $(x + \sqrt{2})(x - \sqrt{2})(x + 1 + \sqrt{3}\,i)(x + 1 - \sqrt{3}\,i)$
zeros: $\{-1 - \sqrt{3}\,i, -1 + \sqrt{3}\,i, -\sqrt{2}, \sqrt{2}\}$

Or, use the two real roots $(x + \sqrt{2})(x - \sqrt{2}) = x^2 - 2$ and do long division to get $x^2 + 2x + 4$, then use the quadratic formula on this factor.

$$x = \frac{-2 \pm \sqrt{2^2 - 16}}{2} = \frac{-2 \pm 2\sqrt{3}\,i}{2} = -1 \pm \sqrt{3}\,i$$

Multiplicity of Zeros

Sometimes the same value for a zero occurs more than once, for example $x^3 + 5x^2 + 3x - 9$ factors to $(x + 3)^2(x - 1)$.

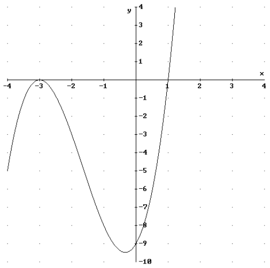
What does this mean for the graph of the polynomial?

The rule is: for a factor $(x - c)^k$

if *k* is **even**, the graph touches the x-axis at *c* but doesn't cross it (turns around)

if *k* is **odd**, the graph crosses the x-axis at *c*

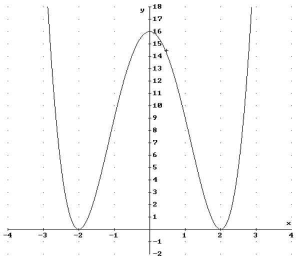
So, for $(x + 3)^2(x - 1)$, the graph will cross the x-axis at 1 but only touch the x-axis at -3 and then turn around.



The polynomial $x^4 - 8x^2 + 16$ factors to:

$(x^2 - 4)^2$ or $(x + 2)^2(x - 2)^2$

Since both factors have even exponents, the graph will touch the x-axis at -2 and 2, but then turn around at both values and not cross.



The polynomial $x^4 - 2x^2$ factors to $x^2(x^2 - 2)$ or $x^2(x + \sqrt{2})(x - \sqrt{2})$.

The graph will touch and turn around at $x = 0$ but cross the x-axis at $\pm\sqrt{2}$.