Complex Roots

For a complex number x + yi, if a + bi is an nth root of x + yi ($a + bi = \sqrt[n]{x + yi}$)
then **$(a + bi)^n = x + yi$**. But we will be working with polar forms of complex numbers, where $x + yi = R(\cos\theta + i\sin\theta)$ and **$[r(\cos\alpha + i\sin\alpha)]^n = R(\cos\theta + i\sin\theta)$**.

Given **$x^4 = -81$**, we are looking for the 4^{th} roots of **-81** (forget the x!).

1) Convert to complex form: $-81 = -81 + 0i$

2) Convert to polar form: $R = \sqrt{(-81)^2 + 0^2} = 81$

$\tan\theta = \dfrac{0}{-81} = 0$ x is negative, so $\theta = 180^o$

$-81 + 0i = 81(\cos 180^o + i\sin 180^o)$

3) So we are looking for a complex number **$r(\cos\alpha + i\sin\alpha)$** such that:

$[r(\cos\alpha + i\sin\alpha)]^4 = 81(\cos 180^o + i\sin 180^o)$

or: $r^4(\cos 4\alpha + i\sin 4\alpha) = 81(\cos 180^o + i\sin 180^o)$

$r^4 = 81$, so $r = 3$

$4\alpha = 180^o + 360^o \bullet k$ $k = 0, 1, 2, 3$

$$\alpha = \frac{180^o + 360^o \bullet k}{4}$$

$\alpha = 45^o, 135^o, 215^o, 315^o$

4) And the 4 roots are:

$3(\cos 45^o + i\sin 45^o) = \dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i$

$3(\cos 135^o + i\sin 135^o) = -\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i$

$3(\cos 225^o + i\sin 225^o) = -\dfrac{3\sqrt{2}}{2} - \dfrac{3\sqrt{2}}{2}i$

$3(\cos 315^o + i\sin 315^o) = \dfrac{3\sqrt{2}}{2} - \dfrac{3\sqrt{2}}{2}i$

Given **$x^3 = -i$**, we are looking for the cube roots of **$-i$** (forget the x!).

1) Convert to complex form: $-i = 0 - i$

2) Convert to polar form: $R = \sqrt{0^2 + (-1)^2} = 1$

$\tan\theta = \frac{-1}{0}$ y is negative, so $\theta = 270^o$

$0 - i = 1(\cos 270^o + i\sin 270^o)$

3) So we are looking for a complex number **r(cosα + isinα)** such that:

$[r(\cos\alpha + i\sin\alpha)]^3 = 1(\cos 270^o + i\sin 270^o)$

or: $r^3(\cos 3\alpha + i\sin 3\alpha) = 1(\cos 270^o + i\sin 270^o)$

$r^3 = 1$, so $r = 1$

$3\alpha = 270^o + 360^o \bullet k$ $k = 0, 1, 2$

$\alpha = 90^o, 210^o, 330^o$

4) And the 3 roots are:

$1(\cos 90^o + i\sin 90^o) = i$

$1(\cos 210^o + i\sin 210^o) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$1(\cos 330^o + i\sin 330^o) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Find the 5th roots of -2 + 2i

1) Convert to polar form: $R = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$

$\tan\theta = \frac{2}{-2} = -1$ y is positive, x is negative (QII), so $\theta = 135^o$

$-2 + 2i = \sqrt{8}\ (\cos 135^o + i\sin 135^o)$

2) $[r(\cos\alpha + i\sin\alpha)]^5 = \sqrt{8}\ (\cos 135^o + i\sin 135^o)$
$r^5(\cos 5\alpha + i\sin 5\alpha) = \sqrt{8}\ (\cos 135^o + i\sin 135^o)$

$r^5 = \sqrt{8}$, so $r = \sqrt[10]{8}$

$5\alpha = 135^o + 360^o \bullet k$ $k = 0, 1, 2, 3, 4$

$\alpha = 27^o, 99^o, 171^o, 243^o, 315^o$

3) The roots are: $\sqrt[10]{8}$ cis 27^o, $\sqrt[10]{8}$ cis 99^o, $\sqrt[10]{8}$ cis 171^o, $\sqrt[10]{8}$ cis 243^o, $\sqrt[10]{8}$ cis 315^o

Solve $x^3 - 27 = 0$

1) $x^3 = 27$ Convert 27 to complex form: $27 = 27 + 0i$

2) Convert to polar form: $R = \sqrt{27^2 + 0^2} = 27$

$\tan\theta = \dfrac{0}{27} = 0$ x is positive, so $\theta = 0^o$

$27 + 0i = 27(\cos 0^o + i\sin 0^o)$

3) So we are looking for a complex number **r(cosα + isinα)** such that:

$[r(\cos\alpha + i\sin\alpha)]^3 = 27(\cos 0^o + i\sin 0^o)$

or: $r^3(\cos 3\alpha + i\sin 3\alpha) = 27(\cos 0^o + i\sin 0^o)$

$r^3 = 27$, so $r = 3$

$3\alpha = 0^o + 360^o \bullet k$ $k = 0, 1, 2$

$\alpha = 0^o + 120^o \bullet k$

$\alpha = 0^o, 120^o, 240^o$

4) The roots are:

$3(\cos 0^o + i\sin 0^o) = 3$

$3(\cos 120^o + i\sin 120^o) = -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2} i$

$3(\cos 240^o + i\sin 240^o) = -\dfrac{3}{2} - \dfrac{3\sqrt{3}}{2} i$

Let's solve this another way:

$x^3 - 27 = (x - 3)(x^2 + 3x + 9) = 0$

One solution is $x = 3$. To find the others, use the quadratic formula.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2} = \frac{-3 \pm \sqrt{-27}}{2} = \frac{-3 \pm 3\sqrt{3}\ i}{2} = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2} i$$

Problems

1. Find the cube roots of 27i.

2. Solve the equation $x^3 + 125 = 0$.

3. Solve the equation $x^4 - (8 + 8i\sqrt{3}) = 0$

4. Find all 4^{th} roots of 1.

Answers

1. Cube roots of 27i are:

$$3(\cos 30^o + i\sin 30^o) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

$$3(\cos 150^o + i\sin 150^o) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

$$3(\cos 270^o + i\sin 270^o) = -3i$$

2. Cube roots of -125 are:

$$5(\cos 60^o + i\sin 60^o) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

$$5(\cos 180^o + i\sin 180^o) = -5$$

$$5(\cos 300^o + i\sin 300^o) = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$$

3. The 4^{th} roots of $(8 + 8i\sqrt{3})$ are:

$2(\cos 15^o + i\sin 15^o)$, $2(\cos 105^o + i\sin 105^o)$, $2(\cos 195^o + i\sin 195^o)$, $2(\cos 285^o + i\sin 285^o)$

4. 4^{th} roots of 1 are:

$$1(\cos 0^o + i\sin 0^o) = 1$$

$$1(\cos 90^o + i\sin 90^o) = i$$

$$1(\cos 180^o + i\sin 180^o) = -1$$

$$1(\cos 270^o + i\sin 270^o) = -i$$