Graphs of Rational Functions: Asymptotes

(MAT 116, MAT 133, MAT 201, MAT 218)

Vertical Asymptotes

$$F(x) = \frac{P(x)}{Q(x)} = \frac{x^2 - x - 12}{x^2 - 2x - 8}$$

Step 1 Factor P(x) and Q(x) completely and cancel any common factors.

$$\frac{x^2 - x - 12}{x^2 - 2x - 8} = \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}(x+2)} = \frac{x+3}{x+2}$$

Step 2 Find the real zeros of the denominator. $x + 2 = 0 \rightarrow x = -2$

Step 3 The vertical asymptotes are at x = real zeros. vertical asymptote at $x = -2$

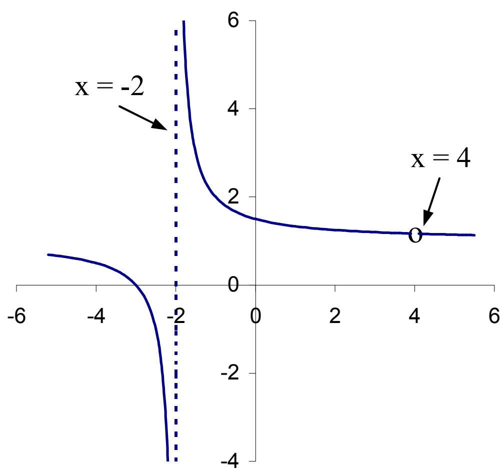
If there are no real zeros, there are no vertical asymptotes.

There is another zero in the denominator of the original function: $x - 4 = 0 \rightarrow x = 4$. This is not an asymptote, but this value of x will produce a zero in the denominator and therefore make the function undefined. When you graph the function, put a "hole" (open circle) in the graph at this point.

When the numerator and denominator of a rational function have common factors, the function will graph just like the reduced form of the function after you cancel out the common factors. However, you still have to account for all points where the original function is undefined.

$$F(x) = \frac{x^2 - x - 12}{x^2 - 2x - 8}$$

(graphs like $\frac{x+3}{x+2}$, except for the "hole")



To find the y value of the "hole", plug an x value of 4 into the reduced function

$$\frac{x+3}{x+2} \rightarrow y = \frac{7}{6}$$

Horizontal Asymptotes

$$F(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_0}$$

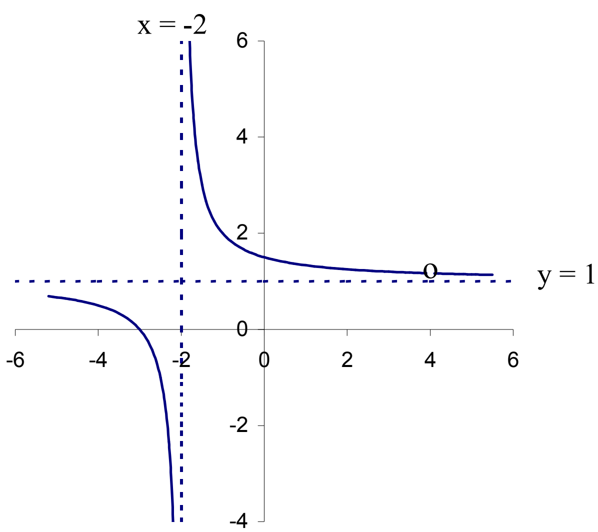
Condition	Description	Asymptote
$n < m$	The highest power of x in the denominator is larger than the highest power in the numerator	the x-axis
$n = m$	The highest power of x in the denominator is the same as the highest power in the numerator	$y = \frac{a_n}{b_m}$
$n > m$	The highest power of x in the numerator is larger than the highest power in the denominator	no horizontal asymptote

$$F(x) = \frac{x^2 - x - 12}{x^2 - 2x - 8}$$

$n = m$

$$y = \frac{a_n}{b_m} = \frac{1}{1} = 1$$

horizontal asymptote at $y = 1$



Slant Asymptotes

$$F(x) = \frac{P(x)}{Q(x)} = \frac{x^3+1}{x^2-1}$$

If the highest power of the variable in the numerator is one higher than the highest power in the denominator, the function graph has a slant asymptote. (Note that if the numerator is of a higher degree than the denominator, it will not have a horizontal asymptote – a function will never have both a horizontal and a slant asymptote.)

Step 1 Factor P(x) and Q(x) completely and cancel any common factors. If there are no common factors, switch back to the unfactored version for Step 2.

$$\frac{P(x)}{Q(x)} = \frac{x^3+1}{x^2-1} = \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}(x-1)} = \frac{x^2-x+1}{x-1}$$

Step 2 Do long division with the result.

$$\begin{array}{r|l} & x + \frac{1}{x-1} \\ x-1 & x^2 - x + 1 \\ & \underline{x^2 - x} \\ & \qquad 1 \end{array}$$

Step 3 Set $y = x + \frac{1}{x-1}$.

As $x \to \pm\infty$, $\frac{1}{x-1} \to 0$

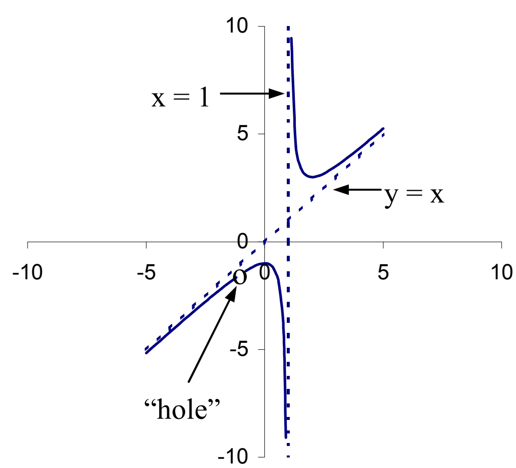
So, you can always drop the last term: $y = x + \cancel{\frac{1}{x-1}} \quad \to \quad y = x$

The slant asymptote is y = x

$$F(x) = \frac{x^3+1}{x^2-1}$$

Slant asymptote at y = x

(Vertical asymptote at x = 1)
(Function undefined at x = -1)



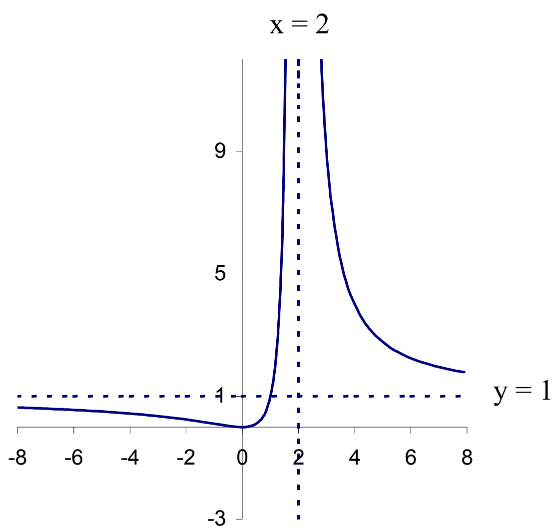
My graph crosses an asymptote! What did I do wrong?!

Although asymptotes often look like some magic boundaries that the graph can never quite reach, sometimes the graph, or a section of the graph, *does* cross an asymptote.

Horizontal and slant asymptotes are lines that the graph gets infinitely close to in the *far right* or *far left* regions of the graph. A horizontal asymptote limits the value of the function at very large values of x (positive or negative). But closer to the origin, the graph may cross either a horizontal or a slant asymptote.

A graph can **never cross a vertical asymptote**, because the function becomes undefined at that x value (an exception would be if you specifically define a function value for that x value).

$$F(x) = \frac{x^2}{x^2 - 4x + 4}$$



$$F(x) = \frac{2x^3 + 5x^2 + 1}{x^2 + x + 3}$$