

Voting Methods (Mat 108)

Several different methods can be used to determine the winner in an election that uses a preference ballot, which voters use to rank all the candidates in order of preference. Each order of preference is counted to produce a preference table. For example, suppose college students are voting for a new mascot and the choices are: Aardvark, Bear, Cougar, and Dog. The students are asked to list the four proposed mascots in order of preference, and the following votes are recorded:

	Number of Votes			
	36	30	13	3
1 st	A	B	C	D
2 nd	B	C	B	A
3 rd	C	A	A	B
4 th	D	D	D	C

We'll look at four ways of determining the winner.

1) The Plurality Method

This method only considers the first place votes. The candidate with the most first place votes wins. The winner is not required to receive a majority of the votes, that is, more than 50%. In the above example, the winner is **Aardvark**, with 36 first place votes (44%).

2) The Borda Count Method

In this method, the places are assigned points: last place gets 1 point, next-to-last gets 2 points, etc. Then the points are multiplied by the number of votes to get the total points for each candidate in each ballot order, and the points are added across the different orders. The one with the most points wins.

In this example, first place gets 4 points, second place gets 3 points, third place gets 2 points, and fourth place gets 1 point. The tally for the first column would be:

	36
1 st 4 points	A: $36 \times 4 = 144$
2 nd 3 points	B: $36 \times 3 = 108$
3 rd 2 points	C: $36 \times 2 = 72$
4 th 1 point	D: $36 \times 1 = 36$

Do this for each column and add up all the points for each candidate.

	Number of Points			
	36	30	13	3
1 st	A: 144	B: 120	C: 52	D: 12
2 nd	B: 108	C: 90	B: 39	A: 9
3 rd	C: 72	A: 60	A: 26	B: 6
4 th	D: 36	D: 30	D: 13	C: 3

A gets: $144 + 60 + 26 + 9 = 239$

B gets: $108 + 120 + 39 + 6 = \boxed{273}$ **Bear wins** by this method.

C gets: $72 + 90 + 52 + 3 = 217$

D gets: $36 + 30 + 13 + 12 = 91$

3) Plurality with Elimination

This method requires a majority, and is also known as "instant run-off" voting, because a majority of votes can be reached without holding additional run-off elections. If no candidate gets a majority of first-place votes (more than 50% of the total) in the initial voting, the candidate with the least number of first-place votes is eliminated and the candidates below the eliminated one in each column are moved up. **The number of votes for each column are not changed.** If there's still no majority, the process is repeated until there is a majority.

In our mascot example, the total number of votes was 82. 50% of 82 is 41. A candidate must have more than 41 first-place votes to win by majority. Since none of them have a majority to start with, we eliminate Dog and move the others up if needed.

	Number of Votes			
	36	30	13	3
1 st	A	B	C	A
2 nd	B	C	B	B
3 rd	C	A	A	C

Now A has 39 first-place votes, but that's still not a majority. So we eliminate Cougar and move others up as needed.

	Number of Votes			
	36	30	13	3
1 st	A	B	B	A
2 nd	B	A	A	B

In this round, A has 39 first-place votes and B has 43 first-place votes, so **Bear** is the winner.

4) Pairwise Comparison Method

This method takes candidates in pairs and determines which one is preferred over the other by the most voters. That candidate gets 1 point. If there is a tie, each candidate gets ½ point. After all the comparisons are made, the points are added up and the one with the most points wins.

The number of comparisons C that must be made is: $C = \frac{n(n-1)}{2}$, where n is the number of

candidates. For our mascots, $C = \frac{4(4-1)}{2} = 6$.

$$\left(\begin{array}{l} \text{A favored over B} \\ = \frac{A}{B} \end{array} \right)$$

$$\frac{A}{B} = 36 + 3 = 39 \quad \frac{B}{A} = 30 + 13 = 43$$

B gets 1 point.

$$\frac{A}{C} = 36 + 3 = 39 \quad \frac{C}{A} = 30 + 13 = 43$$

C gets 1 point.

$$\frac{B}{C} = 36 + 30 + 3 = 69 \quad \frac{C}{B} = 13$$

B gets 1 point.

$$\frac{C}{D} = 36 + 30 + 13 = 79 \quad \frac{D}{C} = 3$$

C gets 1 point.

A has 1 point.

B has 3 points.

C has 2 points.

Bear wins.

	Number of Votes			
	36	30	13	3
1 st	A	B	C	D
2 nd	B	C	B	A
3 rd	C	A	A	B
4 th	D	D	D	C

$$\frac{A}{D} = 36 + 30 + 13 = 79 \quad \frac{D}{A} = 3$$

A gets 1 point.

$$\frac{B}{D} = 36 + 30 + 13 = 69 \quad \frac{D}{B} = 3$$

B gets 1 point.

Flaws of Voting Methods

There are four "fairness criteria" that a fair voting system should meet.

1) The Majority Criterion

If a single candidate receives more than half of the first-place votes, that candidate should be the winner. However, it is possible for someone else to win if the Borda Count Method is used. A candidate that scores consistently high in the preference table can accumulate more points by Borda Count than the one with the actual majority of first-place votes. The other voting methods always satisfy this criterion.

For this preference table, A has a majority.

The Borda Count Method gives C as the winner.

	33	25	7
1st	A	B	C
2nd	C	C	B
3rd	B	A	A

2) The Head-to-Head Criterion

If a candidate is favored over every other candidate when compared separately, then that candidate should win. This is similar to the Pairwise Comparison Method, which will always meet this criterion. All other voting methods can violate the head-to-head criterion.

For example, given the table to the right:

The Plurality Method makes A the winner.

	15	13	3
1st	A	B	C
2nd	B	C	B
3rd	C	A	A

Comparisons are: $\left(A \text{ favored over } B = \frac{A}{B} \right)$

$$\frac{A}{B} = 15 \text{ votes}$$

$$\frac{B}{A} = 16 \text{ votes}$$

B is favored over A.

$$\frac{B}{C} = 38 \text{ votes}$$

$$\frac{C}{B} = 3 \text{ votes}$$

B is favored over C.

B is favored over both A and C, so B wins.

3) The Monotonicity Criterion

If a candidate wins an election, and, in a reelection, the only changes are changes that favor the candidate, then the same candidate should win the reelection. For example, if a candidate wins a straw vote (pre-election poll), then gains additional support in the actual election without losing any original support, that candidate should win the actual election. This criterion can actually be violated by the Plurality with Elimination Method, which can produce a different winner from the straw vote winner.

For example, given the table to the right:

	12	9	8	3
1st	C	B	A	A
2nd	A	C	B	C
3rd	B	A	C	B

Using Plurality with Elimination, C is the winner in the straw vote election.

In the actual election, the 3 votes in the last column were changed to C, A, B, adding to the first column. Plurality with Elimination makes B the winner, even though C actually gained more votes through the switch.

	15	9	8
1st	C	B	A
2nd	A	C	B
3rd	B	A	C

4) The Irrelevant Alternatives Criterion.

If a candidate wins an election and, before the results are announced, one or more candidates withdraw from the running, the same candidate should be the winner on the recount. All four voting methods may violate this criterion.

For example, given the table to the right:

	77	44	22
1st	B	C	A
2nd	C	A	B
3rd	D	D	D
4th	A	B	C

By the Borda Count Method, C wins the election.

If A drops out, the table looks like:

	77	44	22
1st	B	C	B
2nd	C	D	D
3rd	D	B	C

Now, by the Borda Count Method, B wins.

Given the table to the right:

	12	9	4	4
1st	A	V	V	R
2nd	R	R	A	A
3rd	V	A	R	V

By the Plurality Method, V wins the election.

If R drops out, the table looks like:

	12	9	4	4
1st	A	V	V	A
2nd	V	A	A	V

Now by the Plurality Method, A wins the election.

Summary

Criterion →	Majority	Head-to-Head	Monotonicity	Irrelevant Alternatives
Plurality	Always satisfies	May not satisfy	Always satisfies	May not satisfy
Borda	May not satisfy	May not satisfy	Always satisfies	May not satisfy
Plurality with Elimination	Always satisfies	May not satisfy	May not satisfy	May not satisfy
Pairwise Comparison	Always satisfies	Always satisfies	Always satisfies	May not satisfy