Special Factoring Rules (5.4)

Part of this worksheet deals with factoring the special products covered in Chapter 4, and part of it covers factoring some new special products. If you can identify these special products, you can factor them much faster by memorizing a few equations (helpful on a test!).

Procedures in this worksheet assume you have already factored out any common factors.

Factoring Trinomials into Squares of Binomials

In Chapter 4, you learned that $\mathbf{(A+B)^2 = A^2 + 2AB + B^2}$ and $\mathbf{(A-B)^2 = A^2 - 2AB + B^2}$. Now we're going to do the reverse – start with a trinomial and determine if it factors into a binomial square.

The general method is to check the middle term of the trinomial against the first and last terms. Keep the following model in mind – it's the basis for our first comparison:

$$\mathbf{A^2 + 2AB + B^2 = (A + B)^2}$$

Example 1: Factor $\mathbf{x^2 + 12x + 36}$

Step 1
Are the first term and the last term both positive squares? — Yes: $\mathbf{(x)^2}$ and $\mathbf{(6)^2}$
Rewriting them as squares identifies our A and B terms. — $\mathbf{A = x}$ $\mathbf{B = 6}$

Step 2
Now form the product $\mathbf{2 \cdot A \cdot B}$ — $\mathbf{2AB = 2 \cdot x \cdot 6 = 12x}$

Step 3
Is this product the same as the middle term of the trinomial: **12x**? — Yes!

Step 4
Following the model, we can now complete our factoring equation with confidence

$$\mathbf{x^2 + 12x + 36}$$
$$\mathbf{= x^2 + 2 \cdot 6 \cdot x + 6^2 \quad (A^2 + 2AB + B^2)}$$
$$\mathbf{= (x + 6)^2 \quad = (A + B)^2}$$

For the next example, keep this model in mind: $\mathbf{A^2 - 2AB + B^2 = (A - B)^2}$
(only the sign has changed)

Example 2: Factor $\mathbf{9x^2 - 24x + 16}$

Step 1

Are the first term and the last term both positive squares? Rewriting them as squares identifies our A and B terms.

Yes: $\mathbf{3^2x^2 = (3x)^2}$ **and** $\mathbf{(4)^2}$

$\mathbf{A = 3x \quad B = 4}$

Step 2

Important! Notice the **negative** sign in front of the middle term of the trinomial. This fits our second model. Now form the product $\mathbf{-2 \cdot A \cdot B}$

$\mathbf{-2AB = -2 \cdot 3x \cdot 4 = -24x}$

Step 3

Is this product the same as the middle term of the trinomial: **-24x**?

Yes!

Step 4

Following the second model, we can now complete our factoring equation.

$$\begin{aligned} & 9x^2 - 24x + 16 & \\ &= (3x)^2 - 2\cdot3x\cdot4 + 4^2 & (A^2 - 2AB + B^2) \\ &= (3x - 4)^2 & = (A - B)^2 \end{aligned}$$

Before you start doing a longer method of factoring, do a quick check of the trinomial for these things:

1) Are the first and last terms both squares of other terms or numbers?
2) Are the signs on the first and last terms both positive?
3) If I multiply together those terms or numbers that are squared, then multiply the product by 2, do I get the middle term (ignore its sign for now)?
4) If so, I know it's a binomial square, and the sign between terms in the binomial will be the sign in front of the middle term of the trinomial.

	$\mathbf{x^2 + 12x + 36}$ ↑ ↑ **both squares, both positive**	$\mathbf{9x^2 - 24x + 16}$ ↑ ↑ **both squares, both positive**
Rewrite:	$\mathbf{(x)^2 + 12x + (6)^2}$ ↑	$\mathbf{(3x)^2 - 24x + (4)^2}$ ↑
Check:	**middle term** $\mathbf{= 2 \cdot x \cdot 6}$	**middle term** $\mathbf{= 2 \cdot 3x \cdot 4}$
	$\mathbf{(x)^2 + 12x + (6)^2}$ ↑ **sign positive**	$\mathbf{(3x)^2 - 24x + (4)^2}$ ↑ **sign negative**
Factor:	$\mathbf{(x + 6)^2}$ ↑ **sign positive**	$\mathbf{(3x - 4)^2}$ ↑ **sign negative**

Alternative method for factoring $x^2 + bx + c$ or $x^2 - bx + c$

Let's look again at $\mathbf{x^2 + 12x + 36}$ and its complement $\mathbf{x^2 - 12x + 36}$.
If the coefficient in front of the $\mathbf{x^2}$ term is 1, then you can quickly check the last term against the middle.

Step 1
Divide the middle term by 2x (ignore the sign). $\mathbf{12x \div 2x = 6}$

Step 2
Square the result. $\mathbf{6^2 = 36}$

Step 3
Is this the same as the last term in the trinomial? **Yes!**

Step 4
Factor the trinomial as a binomial square. The sign between terms will be the same as the sign on the middle term.

$x^2 + 12x + 36$	$x^2 - 12x + 36$
$= (x + 6)^2$	$= (x - 6)^2$

Factoring Differences of Squares

Another special product in Chapter 4 was the difference of squares:

$$(A + B)(A - B) = A^2 - B^2$$

Let's reverse the process by recognizing when a binomial is a difference of squares and use this model equation: $\mathbf{A^2 - B^2 = (A + B)(A - B)}$

Example 3: Factor $\mathbf{x^2 - 64}$.

Step 1
Are both terms squares and is there a minus sign between them?

= "difference of squares"
↓
Yes: $(x)^2 - (8)^2$
$A = x \quad B = 8$

Step 2
Factor the "difference of squares".

$$x^2 - 64 = (x)^2 - (8)^2 = (x + 8)(x - 8)$$

$$A^2 - B^2 = (A + B)(A - B)$$

EASY, HUH? ☺

OK, sometimes the difference of squares is slightly more complicated to recognize.

Example 4: Factor $\mathbf{625x^2 - 121}$.

Step 1

Are both terms squares and is there a minus sign between them? Use your calculator or a squares table to determine that.

$$625x^2 - 121$$
$$= 25^2x^2 - 11^2$$
$$= (25x)^2 - (11)^2$$
$$A = 25x \quad B = 11$$

Step 2

Factor the "difference of squares".

$$625x^2 - 121 = (25x)^2 - (11)^2 \qquad A^2 - B^2$$
$$= (25x + 11)(25x - 11) \qquad = (A + B)(A - B)$$

Let's throw in some more factors.

Example 5: Factor $\mathbf{81x^2y^2 - 4z^2}$

$$81x^2y^2 - 4z^2$$
$$= 9^2x^2y^2 - 2^2z^2$$
$$= (9xy)^2 - (2z)^2 \qquad A = 9xy,\ B = 2z$$

$$= (9xy + 2z)(9xy - 2z)$$

Remember the Laws of Exponents? You may have to get creative to produce squares!

Example 6: Factor $\mathbf{16x^4 - y^8}$

$$16x^4 - y^8 = (4)^2(x^2)^2 - (y^4)^2$$
$$\uparrow \quad \uparrow \quad \uparrow$$

the important thing is that every component is squared!

$$(4)^2(x^2)^2 - (y^4)^2 = (4x^2)^2 - (y^4)^2$$
$$= (4x^2 + y^4)(4x^2 - y^4)$$

Are we done? Look again at the second term!

$\mathbf{(4x^2 - y^4)}$ looks suspicious. Could it also be a difference of squares?

$$4x^2 - y^4 = (2)^2x^2 - (y^2)^2$$
$$= (2x)^2 - (y^2)^2 \qquad A = 2x,\ B = y^2$$
$$= (2x + y^2)(2x - y^2)$$

So, the complete factoring of $\mathbf{16x^4 - y^8}$ is $\mathbf{(4x^2 + y^4)(2x + y^2)(2x - y^2)}$. It ain't over 'til it's over!

Factoring the Sum of Squares: $A^2 + B^2$

FORGET IT! CAN'T DO IT! NO WAY!

Factoring the Sum or Difference of Cubes

Trust me on this one – just know the equations and how to apply them.

The equation for the sum of cubes is: $\mathbf{A^3 + B^3 = (A + B)(A^2 - AB + B^2)}$

The equation for the difference of cubes is: $\mathbf{A^3 - B^3 = (A - B)(A^2 + AB + B^2)}$

Here's how I remember them:
1) The sign in the first factor is the <u>same sign</u> as between the cubed terms
2) The sign before the middle term of the second factor is the <u>opposite of the sign</u> between the cubed terms.
3) The signs of the first and last terms in the second factor are always positive.
4) Don't confuse the second factor with a binomial square – the middle term is just **AB**, <u>not</u> **2AB!**

$\mathbf{A^3 + B^3 = (A + B)(A^2 - AB + B^2)}$
↓ same ↑ ↑ ↑
↓ opposite | always positive

$\mathbf{A^3 - B^3 = (A - B)(A^2 + AB + B^2)}$
↓ same ↑ ↑ ↑
↓ opposite | always positive

<u>Example 7</u>: Factor $\mathbf{27x^3 + 64}$

Step 1
Are both terms cubes? Rewrite as such.
Identify the **A** and **B** terms.

Yes: $\mathbf{27x^3 + 64 = 3^3x^3 + 4^3 = (3x)^3 + (4)^3}$
$\mathbf{A = 3x \quad B = 4}$

Step 2
Plug the **A** and **B** terms into the model equation.

$$\mathbf{(3x)^3 + 4^3 = (3x + 4)((3x)^2 - (3x)(4) + (4)^2)}$$
$$\mathbf{A^3 + B^3 = (A + B)\ (A^2 \ - \ A \cdot B \ + B^2)}$$

Step 3
Simplify the result.

$$\mathbf{(3x)^3 + 4^3 = (3x + 4)((3x)^2 - (3x)(4) + (4)^2)}$$
$$\mathbf{= (3x + 4)(9x^2 - 12x + 16)}$$
↑
remember opposite sign!

<u>Example 8</u>: Factor $\mathbf{x^3y^3 - 125z^3}$

Step 1
Are both terms cubes? Rewrite as such.
Identify the **A** and **B** terms.

Yes: $\mathbf{x^3y^3 - 125z^3 = (xy)^3 - (5z)^3}$
$\mathbf{A = xy \quad B = 5z}$

Step 2
Plug the **A** and **B** terms into the model equation.

$$\mathbf{(xy)^3 - (5z)^3 = (xy - 5z)((xy)^2 + 5xyz + (5z)^2)}$$
$$\mathbf{A^3 \ - \ B^3 \ = \ (A - B)\ (A^2 \ + \ A \cdot B \ + B^2)}$$

Step 3

Simplify the result.

$$(xy)^3 - (5z)^3 = (xy - 5z)((xy)^2 + 5xyz + (5z)^2)$$
$$= (xy - 5z)(x^2y^2 + 5xyz + 25z^2)$$

↑

remember opposite sign!

You may have to sometimes get creative and produce cubes, just like with the difference of squares.

Consider: $27x^6 - 64y^6$

$$27x^6 - 64y^6 = 3^3(x^2)^3 - 4^3(y^2)^3$$

↑ ↑ ↑ ↑

the important thing is that every component is cubed!

$$= (3x^2)^3 - (4y^2)^3 \qquad A = 3x^2, B = 4y^2$$
$$= (3x^2 - 4y^2)(9x^4 + 12x^2y^2 + 16y^4)$$
$$(A - B) \quad (A^2 + A \cdot B + B^2)$$

Sometimes you may have to mix and match your approach. This is not for the faint of heart! But have courage – the basic guideline is <u>beware of exponents that are divisible by both 2 and 3</u>.

Consider: $64a^6 - 4096b^6$ Is this a difference of squares or a difference of cubes?

a^6 and b^6 can be rewritten in two ways**:** $(a^2)^3$ and $(b^2)^3$ <u>or</u> $(a^3)^2$ and $(b^3)^2$

Also: $64 = 4^3 = (2^2)^3$ <u>or</u> $64 = 8^2 = (2^3)^2$ $\qquad 64 = 2^6$

$4096 = 16^3 = (4^2)^3$ <u>or</u> $4096 = 64^2 = (4^3)^2$ $\qquad 4096 = 4^6$

$64a^6 - 4096b^6$ **factors completely to** $(2a)^6 - (4b)^6$

So, <u>all the components</u> of the two terms have **exponents of 6** – divisible by both 2 and 3, or factorable into either squares or cubes.
Which route do we choose - a difference of squares or a difference of cubes?

WHEN IN DOUBT, START WITH A DIFFERENCE OF SQUARES!!

$$(2a)^6 - (4b)^6 = ((2a)^3)^2 - ((4b)^3)^2 \qquad A = 8a^3 \qquad B = 64b^3$$
$$= (8a^3 + 64b^3)(8a^3 - 64b^3) \qquad (A + B)(A - B)$$

Now we have a <u>sum of cubes</u> and a <u>difference of cubes</u>, <u>both of which are factorable</u>.

$$(8a^3 + 64b^3) = (2a)^3 + (4b)^3$$
$$= (2a + 4b)(4a^2 - 8ab + 16b^2)$$

$$(8a^3 - 64b^3) = (2a)^3 - (4b)^3$$
$$= (2a - 4b)(4a^2 + 8ab + 16b^2)$$

and the total factoring is:

$$64a^6 - 4096b^6 = (2a + 4b)(4a^2 - 8ab + 16b^2)(2a - 4b)(4a^2 + 8ab + 16b^2)$$ Phew!

What if we had initially factored as a difference of cubes?

$$\mathbf{(2a)^6 - (4b)^6 = ((2a)^2)^3 - ((4b)^2)^3} \qquad \mathbf{A = 4a^2} \qquad \mathbf{B = 16b^2}$$
$$\mathbf{= (4a^2 - 16b^2)(16a^4 + 64a^2b^2 + 256b^4)}$$

$\mathbf{(4a^2 - 16b^2)}$ is a difference of squares and can be factored into $\mathbf{(2a + 4b)(2a - 4b)}$, but what do we do with the second factor?

Well, it is factorable, but not in this course! So you would have to stop here and your answer won't match the one in the book.

(Notice that we didn't have a choice with $\mathbf{27x^6 - 64y^6}$ because **27** only factors to $\mathbf{3^3}$.)

Remember, if you have a choice of difference of squares or difference of cubes – <u>on all the components</u> of the terms - START WITH A DIFFERENCE OF SQUARES.

Sample Exercises

Factor:

1. $x^2 + 2x + 1$
2. $a^2 - 14a + 49$
3. $4x^2 + 44x + 121$
4. $x^2y^2 + 20xy + 100$
5. $49x^2 - 56x + 16$
6. $25x^2 - 90x + 81$
7. $p^2 + 16pq + 64q^2$
8. $9x^2y^2 + 12xyz + 4z^2$

Factor:

9. $x^2 - 169$
10. $36 - a^2b^2$
11. $9m^2 - 49n^2$
12. $4x^{10} - 25y^6$
13. $-a^2 + \frac{1}{49}b^2$
14. $16x^2 + 9$

Factor:

15. $x^3 + 27$
16. $8a^3 - 1$
17. $x^3y^6 - z^3$
18. $\frac{1}{27} + 125x^3$
19. $a^6 + b^6$
20. $x^{3a} - \frac{8}{27}y^{3a}$

Answers

1. $(x + 1)^2$
2. $(a - 7)^2$
3. $(2x + 11)^2$
4. $(xy + 10)^2$
5. $(7x - 4)^2$
6. $(5x - 9)^2$
7. $(p + 8q)^2$
8. $(3xy + 2z)^2$
9. $(x + 13)(x - 13)$
10. $(6 + ab)(6 - ab)$
11. $(3m + 7n)(3m - 7n)$
12. $(2x^5 + 5y^3)(2x^5 - 5y^3)$
13. $(\frac{1}{7}b + a)(\frac{1}{7}b - a)$
14. Bzzt! Not a difference of squares.
15. $(x + 3)(x^2 - 3x + 9)$
16. $(2a - 1)(4a^2 + 2a + 1)$
17. $(xy^2 - z)(x^2y^4 + xy^2z + z^2)$
18. $(\frac{1}{3} + 5x)(\frac{1}{9} - \frac{5}{3}x + 25x^2)$
19. $(a^2 + b^2)(a^4 - a^2b^2 + b^4)$
20. $(x^a - \frac{2}{3}y^a)(x^{2a} + \frac{2}{3}x^ay^a + \frac{4}{9}y^{2a})$

□