23 **PROBLEM SOLVING:** **(6.6, 6.7)**

Rate-Time Problems, Proportions, Solving Formulas for a Letter

Rate-Time Problems

Rate-time problems usually have to do with either distance traveled or work performed.

The basic equations are: $d = r \cdot t$ (distance = rate x time)
or $1 = r \cdot t$ (1 job or quantity = rate x time)

The approach to these problems is to: 1) set up a distance (or quantity)-rate-time table;
2) decide which variable you need to solve for and rewrite the other variable in terms of it; 3) look for an equation, based on the information you have, that will enable you to solve for the desired variable.

Example 1

One car travels 25 km/h faster than another. In the same time that one goes 120 km, the other goes 220 km. Find their speeds.

Step 1. Set up a table and translate the words:

"One car travels 25 km/h faster than another"

Let **r** = the rate of the slow car and
r + 25 = the rate of the other car.

	d =	**r •**	**t**
Slow car		r	
Fast car		r + 25	

"In the same time that one goes 120 km, the other goes 220 km"

The slower car must have gone the 120 km (in the same time)

	d =	**r •**	**t**
Slow car	120	r	
Fast car	220	r + 25	

Step 2. Which variable are we solving for? ***"Find their speeds."*** In this case, it's **r**, the speed. We'll write the time **t** in terms of **r**:

$\mathbf{d = r \cdot t \;\rightarrow\; t = d/r}$

Slow car: $\mathbf{t_1 = 120/r}$
Fast car: $\mathbf{t_2 = 220/(r+25)}$

	d =	**r •**	**t**
Slow car	120	r	120/r
Fast car	220	r+25	220/(r+25)

Step 3. Where's the equation? *"In the same time…"* The times are equal! Set the two expressions for time (in terms of rate) equal to each other:

$$t_1 = t_2 \rightarrow \frac{120}{r} = \frac{220}{r+25}$$

And solve

120 (r + 25) = 220 r
120 r + 3000 = 220 r
3000 = 100 r
r = 30 km/h
r + 25 = 55 km/h

Example 2

In a **work** problem, rates add together – **do not add times or quantities!**

Billy can wreck a truck with a sledge hammer in 3 hours, and Thelma can do it in 5 hrs. How long would it take both of them working together to wreck a truck?

Step 1. Set up a table and fill in the data. In this case, we'll use **Q** for "quantity". Quantity is one job (or one water tank filled, etc) and doesn't change throughout the table.

	Q =	**r •**	**t**
Billy	1		3
Thelma	1		5
Together	1		t

Step 2. Which variable are we solving for? *"How long would it take…"* We need to find **t**, the time it takes to do the job together. So, write **r** in terms of **t**, and add the rates together:

Q = r • t → r = Q/t

Billy: r = 1/3
Thelma: r = 1/5

	Q =	**r •**	**t**
Billy	1	1/3	3
Thelma	1	1/5	5
Together	1	1/3 + 1/5	t

Step 3. Where's the equation? In this example, it's just the original equation Q = r • t, but we use the values in the last line of the table to solve for **t**:

$$\mathbf{Q = r \cdot t}$$

$$1 = (1/3 + 1/5)\ t$$
$$1 = (8/15)\ t$$
$$t = \frac{15}{8} = 1\frac{7}{8}\ \text{hr}$$

Quick work!

Example 3

In distance-rate-time problems, you can add distances and times together.

The speed of an Amazonian water moccasin in still water is 8 mph. The snake swims 12 mi upstream and 12 mi downstream in a total time of 4 hr. What is the speed of the current?

Step 1. Set up a table and translate the words:

A snake travels upstream against the current, so its *overall* speed will be its speed in still water minus the speed of the current. Going downstream, the reverse is true.

Let **r** = the speed of the current:

Upstream speed of snake = 8 – r
Downstream speed of snake = 8 + r

	d =	**r •**	**t**
Upstream	12	8 - r	
Downstream	12	8 + r	
Total			4

Step 2. Which variable are we solving for? *"What is the speed of the current?"* We're solving for **r**, so we'll write **t** in terms of **r**:

d = r • t → t = d/r

Upstream: t_1 = 12/(8-r)
Downstream: t_2 = 12/(8+r)

	d =	**r •**	**t**
Upstream	12	8 - r	12/(8-r)
Downstream	12	8 + r	12/(8+r)
Total	24		4

Step 3. Where's the equation? *"...a total time of 4 hr."* The times add together for a total of 4 hr.

$$t_1 + t_2 = 4$$

$$\frac{12}{8-r} + \frac{12}{8+r} = 4$$

$12(8 + r) + 12(8 - r) = 4(64 - r^2)$
$96 + 12r + 96 - 12r = 256 - 4r^2$
$192 = 256 - 4r^2$
$4r^2 = 64$
$r^2 = 16$
r = 4 mph

Example 4

At 2 am, James Bond escapes the secret island of the evil Dr No with a jet pack that goes 50 mph and heads to home base 200 miles away. An hour later, Dr No's henchman Boris leaves to pursue him with a jet pack that goes 70 mph. At what time will Boris catch up to Bond? Will Bond make it to home base? How will it end?

Step 1. Set up a table and translate the words:

"An hour later..." When Boris catches up to Bond, he will have traveled one hour less than Bond.

Let **t** = the time that Bond travels and
t – 1 = the time that Boris travels

	d =	**r •**	**t**
Bond		50	t
Boris		70	t-1

Step 2. Which variable are we solving for? *"At what time...?"* We are solving for time, so let's write distance in terms of time.

d = r • t

for Bond: $d_1 = 50 \cdot t$
for Boris: $d_2 = 70 \cdot (t-1)$

	d =	**r •**	**t**
Bond	50t	50	t
Boris	70(t-1)	70	t-1

Step 3. Where's the equation? *"…catch up to Bond"* When Boris meets Bond, they will have traveled the <u>same distance</u> – right? So, <u>the distances are equal</u>.

$$\mathbf{d_1 = d_2}$$

$$50t = 70\,(t-1)$$
$$50t = 70t - 70$$
$$20t = 70$$
$$t = 7/2 \text{ or } 3\tfrac{1}{2} \text{ hrs}$$

"At what time will Boris catch up to Bond?"
Since Bond left at 2 am, and it took 3½ hr <u>since that time</u> for Boris to catch him (remember, t was Bond's time), they meet at 5:30 am, just as dawn is breaking.

"Will Bond make it to home base?" Well, let's see. If Bond traveled at 50 mph for 3½ hr, then the distance he traveled was:

$$\mathbf{d = r \bullet t}$$
$$d = 50 \bullet 3\tfrac{1}{2}$$
$$d = 175 \text{ miles}$$

Oops! He's 25 miles short of home base, which was 200 miles away, and is now fighting Boris over open ocean.

"How will it end?" Bond comes *so close* to being defeated by Boris, when, at the crucial instant, Boris' jet pack runs out of gas and he plummets into the shark-infested waters. James Bond lives to fight another day (of course!).

Proportions

If two <u>ratios</u> are equal to each other, the equation is called a <u>proportion</u> and the numbers in the ratios are <u>proportional</u> to each other.

<u>Example 1</u> If I tell you that 2 bags of apples weigh 5 lbs, how would you figure out how much 5 bags of apples would weigh? We can do it by setting up the following equation:

$$\frac{2 \text{ bags}}{5 \text{ lbs}} = \frac{5 \text{ bags}}{x \text{ lbs}}$$

This equation says that the number of bags is *proportional* to their total weight and vice-versa.

To solve, *cross-multiply* to remove the denominators:

$$(2 \text{ bags})(x \text{ lbs}) = (5 \text{ bags})(5 \text{ lbs})$$

or $2x = 5 \bullet 5 = 25$

$$x = 25/2 = 12\tfrac{1}{2} \text{ lbs} = \text{ weight of 5 bags}$$

Example 2   I trap 25 of the chipmunks that live under my lawn and attach tiny flea collars to them, then release them to munch on my bulbs. Later I trap 10 chipmunks and 2 of them are wearing flea collars. How many chipmunks live under my lawn?

The 25 chipmunks with collars are the total number of tagged animals in the total population under my lawn. I assert that the ratio of 2 tagged chipmunks to 10 captured represents the same ratio of 25 tagged chipmunks to the total population.

The proportion equation would be: $$\frac{\text{25 tagged chipmunks}}{\text{Population}} = \frac{\text{2 tagged chipmunks}}{\text{10 trapped chipmunks}}$$

or $$\frac{25}{P} = \frac{2}{10}$$

Cross-multiply and solve:

$25 \cdot 10 = 2 \cdot P$

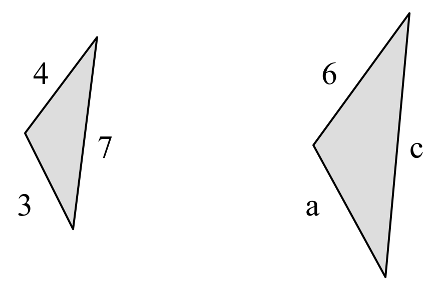
$250 = 2P$

$P = 125$ chipmunks   Probably about right!

Example 3

Similar triangles are triangles that have the same angles. In similar triangles the lengths of the sides in one triangle are proportional to the lengths of the sides in the other triangle.

The following are similar triangles:



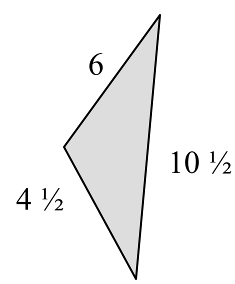
The ratios of corresponding sides are equal:

$$\frac{6}{4} = \frac{a}{3} = \frac{c}{7}$$

To solve for a and c, use the equations:

$$\frac{6}{4} = \frac{a}{3} \quad \text{and} \quad \frac{6}{4} = \frac{c}{7}$$

$3 \cdot 6 = 4 \cdot a$	$7 \cdot 6 = 4 \cdot c$
$18 = 4a$	$42 = 4c$
$a = 4\frac{1}{2}$	$c = 10\frac{1}{2}$



Formulas – Solving for a Letter

Formulas are often in a form that consists mostly or completely of letters (variables). For example, the formula for the distance an object falls, as a function of time, is given by:

Example 1 $\mathbf{s = \frac{1}{2} g t^2}$ (s = distance, g = gravitational constant, t = time)

Let's solve the equation for **g**. Don't let it throw you that you're dealing with letters – just multiply or divide or whatever in the same way you would deal with numbers.

Remember, to <u>solve for a variable</u> means to <u>isolate it</u> on one side of the equation – get rid of everything else on that side of the equation by moving it to the other side.

$$\mathbf{s = \tfrac{1}{2}\, g\, t^2}$$

Clear the fraction: $2 \bullet s = \tfrac{1}{2} \bullet 2 \bullet gt^2$

$2s = gt^2$

Divide to isolate g: $\dfrac{2s}{t^2} = \dfrac{gt^2}{t^2}$

$$\mathbf{g = 2s/t^2}$$

If you have <u>more than one term that contains the variable</u> you are solving for, you must:
1) manipulate the equation until you have all the terms with that variable on one side of the equation; 2) factor out that variable; 3) then multiply or divide to solve for that variable.

<u>Example 2</u> Solve $K = \dfrac{ma - b}{a}$ for a

Clear the fraction: $Ka = ma - b$

Gather the "a" terms: $Ka - ma = -b$

Factor: $a\,(K - m) = -b$

Divide: $a = \dfrac{-b}{K - m}$ or $\dfrac{b}{m - K}$

<u>Example 3</u> Solve $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ for R

Multiply by the LCD $\mathbf{Rr_1r_2}$: $\dfrac{Rr_1r_2}{R} = \dfrac{Rr_1r_2}{r_1} + \dfrac{Rr_1r_2}{r_2}$

$r_1r_2 = Rr_2 + Rr_1$

The R terms are already on the same side, so factor out the R: $r_1r_2 = R\,(r_2 + r_1)$

Divide: $R = \dfrac{r_1r_2}{r_2 + r_1}$

Problems

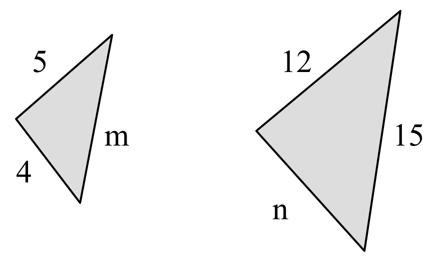
1. One pipe can fill a shark pool in 12 hours and another pipe can fill the pool in 6 hours. How long would it take to fill the pool if both pipes are running?

2. Georgio bicycles 10 mph slower than do Click and Clack on their tandem. Georgio pedals 30 miles in the same time that Click and Clack pedal 50 miles. What are the speeds of both bicycles?

3. François and Fou-Fou cross-country ski at the same rate. After 40 km, François stops to take photographs and Fou-Fou continues. Later, François travels another 20 km, which takes him half the time it took him to go 40 km, and stops again for lunch. Fou-Fou, who never ever stops, travels a total of 4 hours more than François' total travel time, and covers a total distance of 100 km. How long did François travel before stopping the first time?

4. Letitia can work 5 math problems in her head in 45 minutes. How long would it take her to work 25 math problems in her head?

5. Find the unknown sides of the similar triangles:



6. A 15-foot pole casts a 10-foot shadow while a nearby building casts a 42-foot shadow. What is the height of the building?

7. Solve $f = \dfrac{kMm}{d^2}$ for k.

8. Solve $ab + c = ac$ for a.

9. Solve $S = \dfrac{xy - b}{b - x}$ for x.

10. Solve $P = 2(L + W)$ for W.

Answers

1. 4 hours

2. 15 mph, 25 mph

3. 4 hours

	d =	r •	t
François	40 + 20	r	t + t/2
Fou-Fou	100	r	t + t/2 + 4

4. 225 minutes = 3 hours, 45 min.

5. m = 6.25, n = 9.6

6. 63 feet

7. $k = \dfrac{fd^2}{Mm}$

8. $a = \dfrac{c}{c - b}$ or $\dfrac{-c}{b - c}$

9. $x = \dfrac{b(S + 1)}{S + y}$

10. $W = \dfrac{P - 2L}{2}$

Bonus problem: A truck travels in the same direction as a nearby stream, while a snowmobile travels in the opposite direction on the opposite bank. If the truck travels at 4 times the speed of the current, and the snowmobile travels at 1/3 the speed of the truck, and a freight train is speeding toward the river bridge from the east at 70 mph, and a boat on the river left one hour after the snowmobile, on which side of the river will they bury the crash survivors? (*Just kidding!*) ☐