10 Solving Linear Inequalities (Including Word Problems) (2.8)

An inequality statement can be read from left to right or from right to left. (We usually use the variable as our starting point.) The "larger" end of the symbol faces the larger number and the "smaller" end faces the smaller number.

Example

$x < 5$ can be read "x is less than 5" or "5 is greater than x". (Since we are usually interested in the variable, the first expression is more common.)

$-3 \geq y$ is read "y is less than or equal to -3" (or "-3 is greater than or equal to y").

A number is a "solution" to an inequality if, when the variable is replaced with the number, a true statement results.

Example

Is $\frac{1}{2}$ a solution of $x \leq 1$?

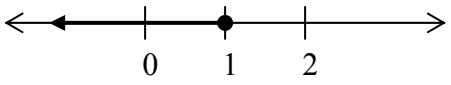
Replace x with $\frac{1}{2}$. Is $\frac{1}{2} \leq 1$ a true statement? Is $\frac{1}{2}$ less than or equal to 1? Yes, $\frac{1}{2}$ is less than 1, so it is a solution. (Don't let the "equal" part of the symbol confuse you. Any number that is less than 1 **or** equal to 1 is a solution.) Notice that this inequality has **many** solutions; there are many numbers that are less than or equal to 1. This is true for most inequalities.

The solutions to an inequality can be "graphed" on a number line.

Example

Graph all the solutions to $x \leq 1$.

Draw a number line. Show several numbers for reference. Shade the number line to the **left** of 1 (that's where all the numbers less than 1 are!). Also, put a large, filled in dot right on the 1 to show that it is also a solution (x can be less than 1 or equal to 1).

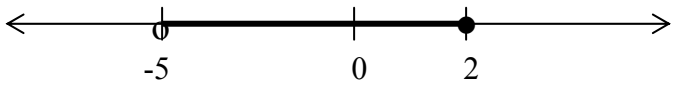


Example

Graph all solutions to $-5 < x \leq 2$.

This kind of inequality (with the variable in the middle between 2 numbers) is called a "compound inequality". It can be read several ways:
"x is greater than –5 AND x is less than or equal to 2"
"-5 is less than x AND x is less than or equal to 2"
"2 is greater than or equal to x AND x is greater than -5"

You can think of x as being "between" the two numbers -5 and 2. To graph this inequality, notice that x must be greater than -5 (which means that -5 is NOT a solution and shouldn't be included in the graph) AND x must be less than or equal to 2 (which means that 2 IS a solution and should be included).



We shade the numbers "between" -5 and 2 using a filled in dot on the 2 (to show it is included) and an open circle around the -5 (to show that it is NOT included).

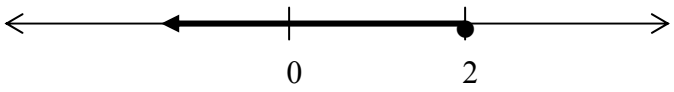
Solving Inequalities

Solving an inequality is just like solving an equation WITH ONE IMPORTANT EXCEPTION. IF YOU **MULTIPLY** OR **DIVIDE** BOTH SIDES OF AN INEQUALITY BY A **NEGATIVE** NUMBER, YOU MUST <u>REVERSE THE DIRECTION</u> OF THE INEQUALITY.

Example

$3x + 2 \leq 8$	Just like in an equation, we want to isolate the variable. Start by adding or subtracting terms to isolate the term with the x in it. Subtract 2 from both sides.
$3x \leq 6$	Now isolate x by dividing both sides by its coefficient 3.
$x \leq 2$	We have now isolated x and solved the inequality.

Solutions to inequalities can be represented in many ways. One way is graphing.



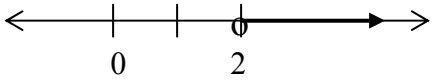
Another common way is by using "set builder notation". Set builder notation consists of set brackets (the "curly" ones), the variable symbol, and then a "description" of the members of the solution set using an inequality or some other description.

Example

$\{x \mid x \le 2\}$ is the "solution set" for the above inequality. It is read "the set of all numbers x such that x is less than or equal to 2"

{	x	|	$x \le 2$	}
↑	↑	↑	↑	↑
set	all numbers	such that	description	set

Example

$4x - 1 < 6x - 5$	There are terms with x's in them on both sides of the inequality. Subtract 6x from both sides (or subtract 4x from both sides).
$-2x - 1 < -5$	Now add 1 to both sides to isolate the term with the x in it.
$-2x < -4$ ↑ **reverse** ↓	To isolate x we must <u>divide both sides</u> by its coefficient **-2**. Here we run into the IMPORTANT EXCEPTION discussed earlier. We are dividing both sides of the inequality by a <u>negative</u> number. We must REVERSE the direction of the inequality.
$\frac{-2x}{-2} > \frac{-4}{-2}$	
	Simplify both sides.
$x > 2$	The solution set is $\{ x \mid x > 2 \}$ The graph of the solution set is 

NOTE: You may use all the same techniques you used for clearing fractions, decimals, and parentheses from equations when you are solving inequalities. Just remember that if you multiply or divide both sides by a negative number, you must change the direction of the inequality symbol.

Example

$\frac{3t}{4} - 2\left(t + \frac{1}{3}\right) \ge 4t - \frac{3}{2}$ Use distributive law to clear parentheses.

$\frac{3t}{4} - 2t - \frac{2}{3} \ge 4t - \frac{3}{2}$ Clear fractions by multiplying both sides by the least common multiple for all the denominators (which is 12). We are multiplying by a *positive* number. DO NOT change direction of inequality!

$12\left(\frac{3t}{4} - 2t - \frac{2}{3}\right) \ge 12\left(4t - \frac{3}{2}\right)$

$12 \cdot \frac{3t}{4} - 12 \cdot 2t - 12 \cdot \frac{2}{3} \ge 12 \cdot 4t - 12 \cdot \frac{3}{2}$

$9t - 24t - 8 \ge 48t - 18$ Collect like terms.

$-15t - 8 \geq 48t - 18$

$-48t \qquad -48t$ Subtract 48t from both sides (or add 15t to both sides).

$-63t - 8 \geq -18$

$+8 \quad +8$ Add 8 to both sides.

$-63t \geq -10$

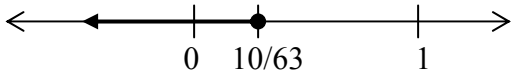
$\frac{-63t}{-63} \leq \frac{-10}{-63}$ Divide both sides by **-63** (reverse inequality!)

$t \leq \frac{10}{63}$

Note: If you chose to add 15t to both sides, you would not have had to divide both sides by a negative. You would have ended up with $\frac{10}{63} \geq t$. This is the same answer as above. Both answers can be read "t is less than or equal to $\frac{10}{63}$ ".

The solution set for this inequality is $\{ t \mid t \leq \frac{10}{63} \}$

The graph of the solution set is



Note: It is not possible to "check" an inequality. You could not replace t with every number less than or equal to $\frac{10}{63}$. If you wish though, you can do a partial check by picking at least one number that is less than $\frac{10}{63}$ and substituting it into the original inequality to see if you get a true statement.

WORD PROBLEMS

Everything you learned before about word problems also applies to word problems involving inequalities. The tricky part about word problems involving inequalities is learning to translate expressions like "at least, at most, cannot exceed, must exceed, less than, no less than, more than, no more than, is between" into appropriate inequalities facing the right direction! Sometimes it helps to concentrate on the inequality symbol first when setting up the problem and just using words to write the inequality. Then replace your word phrases with the appropriate expressions.

Example

Your quiz grades are 78, 72, 87, and 90. What score on the fifth quiz will make your average quiz grade at least 82?

First concentrate on the part "average quiz grade must be <u>at least 82</u>." "At least 82" means your grade must be <u>82 or higher</u>.

Write down: average grade ≥ 82

Once the inequality is written, work on replacing phrases with mathematical expressions.

Let x = fifth quiz score

Average quiz grade will be the sum of all the scores divided by the number of quizzes: $\dfrac{78+72+87+90+x}{5}$

Average grade ≥ 82 $\qquad \dfrac{78+72+87+90+x}{5} \geq 82$

Solve for x.

$\dfrac{327+x}{5} \geq 82$ Multiply both sides by 5 (a <u>positive</u> number - do not reverse inequality).

$327 + x \geq 410$

$x \geq 83$ Answer the original question with a sentence.
"The fifth quiz grade must be greater than or equal to 83."

Example

You rent a car and are offered 2 payment options. You can pay \$25 a day plus 15¢ a mile (option A) or you can pay \$10 a day plus 40¢ a mile (option B). For what amount of daily miles will option A be the cheaper plan?

We want: Cost of plan A < Cost of plan B Let x = the number of miles driven daily

$$\begin{array}{rcl} 25 + .15x & < & 10 + .40x \\ -.15x & & -.15x \\ \hline 25 & < & 10 + .25x \\ -10 & & -10 \\ \hline 15 & < & .25x \end{array}$$

$$\frac{15}{.25} < \frac{.25x}{.25}$$

$60 < x$
$x > 60$

Plan A will be the cheaper plan if you drive more than 60 miles per day.

PROBLEMS

1. Is 5 a solution to $-3x + 2 \leq 11$?

2. Graph this solution set on a number line: $-2 \leq x < 4$

Solve each inequality. Graph the solution set and also write the solution set using set-builder notation.

3. $-2x + 3 \leq 5$

4. $-3x + 7 > 5 - x$

5. $\frac{2}{3}x - \frac{1}{6} \leq x - \frac{1}{3}$

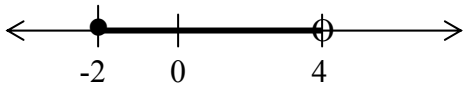
6. $7(-2x - 3) + 2x > -2(-8 + 3x)$

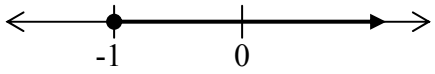
7. To get a grade of C in your course, you must average at least 75% on 4 exams. You have taken the first three exams and gotten scores of 68, 78, and 81. What must you score on the last exam to get a C or better?

8. You have just been given a new job in sales. You have two salary options. You can receive a straight salary of $500 per week (no commission option) or you can receive a salary of $200 per week plus 5% of your weekly sales (commission option). What dollar amount of product must you sell each week in order for the commission option to be the better deal?

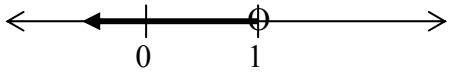
Answers

1. Yes

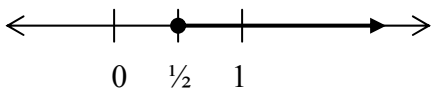
2. 

3. 

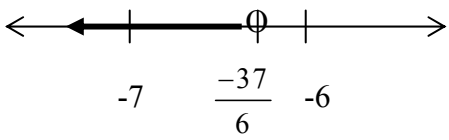
$\{ x \mid x \geq -1 \}$

4. 

$\{ x \mid x < 1 \}$

5. 

$\{ x \mid x \geq \frac{1}{2} \}$

6. 

$\{ x \mid x < \frac{-37}{6} \}$

7. You must get at least 73.

8. You must sell more than $6000 each week.